THE RESOLVENT KERNEL FOR PCF SELF-SIMILAR FRACTALS

MARIUS IONESCU, ERIN P. J. PEARSE, LUKE G. ROGERS, HUO-JUN RUAN,
AND ROBERT S. STRICHARTZ

Abstract. For the Laplacian Δ defined on a p.c.f. self-similar fractal, we give an explicit formula for the resolvent kernel of the Laplacian with Dirichlet boundary conditions, and also with Neumann boundary conditions. That is, we construct a symmetric function $G^{(\lambda)}$ which solves $(\lambda\mathbb{I} - \Delta)^{-1} f(x) = \int G^{(\lambda)}(x, y) f(y)\, d\mu(y)$. The method is similar to Kigami's construction of the Green kernel in [Kig01, §3.5] and is expressed as a sum of scaled and "translated" copies of a certain function $\psi^{(\lambda)}$ which may be considered as a fundamental solution of the resolvent equation. Examples of the explicit resolvent kernel formula are given for the unit interval, standard Sierpinski gasket, and the level-3 Sierpinski gasket SG_3.

Contents

1. Introduction

A theory of analysis on certain self-similar fractals is developed around the Laplace operator Δ in [Kig01]. In this paper, we consider the resolvent function $(\lambda\mathbb{I} - \Delta)^{-1}$ and obtain a kernel for this function when the Laplacian is taken to have Dirichlet or Neumann boundary conditions. That is, we construct a symmetric function $G^{(\lambda)}$ which weakly solves $(\lambda\mathbb{I} - \Delta)G^{(\lambda)}(x, y) = \delta(x, y)$, meaning that

$$\int G^{(\lambda)}(x, y) f(y)\, d\mu(y) = (\lambda\mathbb{I} - \Delta)^{-1} f(x). \tag{1.1}$$

Date: **April 30, 2009.**
2000 *Mathematics Subject Classification.* Primary: 28A80, 35P99, 47A75. Secondary: 39A12, 39A70, 47B39.
Key words and phrases. Dirichlet form, graph energy, discrete potential theory, discrete Laplace operator, graph Laplacian, eigenvalue, resolvent formula, post-critically finite, self-similar, fractal.
The work of EPJP was partially supported by the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.
The work of HJR was partially supported by grant NSFC 10601049, and by the Future Academic Star project of Zhejiang University.
The work of RSS was partially supported by NSF grant DMS 0652440.

For the case $\lambda = 0$, this is just the Green function for Δ. Consequently, it is not surprising that our construction is quite analogous to that of the Green function as carried out in [Kig01, §3.5]; see also [Str06, §2.6] for the case of the Sierpinski gasket (and the unit interval) worked out in detail, and [Kig03].

We present our main results in §1.2, just after the introduction of the necessary technical terms in §1.1. It is the authors' hopes that the resolvent kernel will provide an alternate route to obtaining heat kernel estimates (see [FHK94, HK99]) in this setting, as well as other information about spectral operators of the form

$$\xi(\Delta) = \int_\Gamma \xi(\lambda)(\lambda\mathbb{I} - \Delta)^{-1}\, d\lambda, \tag{1.2}$$

in the same manner as used by Seeley [See67, See69] for the Euclidean situation. Some initial results in this direction will appear in [Rog08].

To explain the method of construction for the resolvent kernel, we carry out the procedure in the case of the unit interval in §2; we believe this particular method has not previously appeared in the literature. In §3, we show how the construction may be generalized to any post-critically finite self-similar fractal. In §5, we give the explicit formulas for the Sierpinski gasket and in §6 we give the explicit formulas for a variant of the Sierpinski gasket which we call SG_3.

Acknowledgements. The authors are indebted to the referee for many keen observations and a scrupulously detailed report.

1.1. **Background, notation, and fundamentals.** We work in the context of post-critically finite (p.c.f.) self-similar fractals. The full and precise definition may be found in [Kig01, Def. 1.3.13], but for the present context it suffices to think of such objects as fractals which may be approximated by a sequence of graphs, via an iterated function system (IFS). A more general setting is possible; cf. [Kig03]. We now make this more precise.

Definition 1.1. Let $\{F_1, F_2, \ldots, F_J\}$ be a collection of Lipschitz continuous functions on $\mathbb{R}^d$ with $0 < Lip(F_j) < 1$ for each j. Let X denote the *attractor* of this IFS, that is, X is the unique nonempty and compact fixed point of the set mapping $F(A) := \bigcup_{j=1}^J F_j(A)$. The set X is frequently also called the *self-similar set* associated to this IFS; existence and uniqueness of X was shown in [Hut81].

From the IFS introduced in the previous definition, we now build a sequence of graphs which approximates X in a suitable sense.

Definition 1.2. We use $\omega = \omega_1\omega_2\cdots\omega_m$ to denote a *word* of length $|\omega| = m$ on the symbol alphabet $\{1, 2, \ldots, J\}$. This notation is used to denote a composition of the mappings F_j via $F_\omega = F_{\omega_1} \circ F_{\omega_2} \circ \ldots \circ F_{\omega_m}$. Similarly, $K_\omega = F_\omega(X)$ refers to a certain *m-level cell*. The collection of all finite words is denoted $W_* := \bigcup_m \{1, 2, \ldots, J\}^m$.

Definition 1.3. Each map F_j of the IFS defining X has a unique fixed point x_j. The *boundary* of X is the largest subset $V_0 \subseteq \{x_1, \ldots, x_N\}$ satisfying

$$F_\omega(X) \cap F_{\omega'}(X) \subseteq F_\omega(V_0) \cap F_{\omega'}(V_0), \qquad \text{for any } \omega \neq \omega' \text{ with } |\omega| = |\omega'|.$$

The p.c.f. condition mentioned above means that cells $F_j(X)$ intersect only at points of $F_j(V_0)$. The *boundary of an m-cell* is $\partial K_\omega := F_\omega(V_0)$.

Let G_0 be the complete graph on V_0, and inductively define $G_m := F(G_{m-1})$. Also, we use the notation $x \sim_m y$ to indicate that x and y are *m-level neighbours*, i.e., that there is an edge in G_m with endpoints x and y. We use $V_m = F^m(V_0)$ to denote the vertices of G_m,

and $V_* := \bigcup_m V_m$. The fractal X is the closure of V_* with respect to either the Euclidean or resistance metric. A discussion of the resistance metric may be found in [Str06, §1.6] or [Kig01, §2.3].

Now we are able to make precise the sense in which X is the limit of graphs: one may compute the Laplacian (and other analytic objects, including graph energy, resistance distance, etc.) for functions $u : X \to \mathbb{R}$ by computing it on G_m and taking the limit as $m \to \infty$.

Definition 1.4. We assume the existence of a *self-similar (Dirichlet) energy form* $\mathcal{E}$ on X. That is, for functions $u : X \to \mathbb{R}$, one has

$$\mathcal{E}(u) = \sum_{j=1}^{J} r_j^{-1} \mathcal{E}(u \circ F_j), \tag{1.3}$$

for some choice of *renormalization factors* $r_1, \ldots, r_J \in (0,1)$ depending on the IFS. This quadratic form is obtained from the approximating graphs as the appropriately renormalized limit of $\mathcal{E}_{G_m}(u) := \mathcal{E}_{G_m}(u,u)$, where the m-level bilinear form is defined

$$\mathcal{E}_{G_m}(u,v) := \frac{1}{2} \sum_{\substack{x,y \in V_m \\ x \underset{m}{\sim} y}} c_{xy}(u(x) - u(y))(v(x) - v(y)). \tag{1.4}$$

The constant $c_{xy} = c_{xy}^{(m)}$ refers to the *conductance* of the edge in G_m connecting x to y (with $c_{xy} = 0$ if there is no such edge). The dependence of $c_{xy}^{(m)}$ on m is typically suppressed, as $x \underset{m}{\sim} y$ for at most one value of m on p.c.f. fractals. The *domain* of $\mathcal{E}$ is

$$\operatorname{dom} \mathcal{E} := \{u : X \to \mathbb{R} \colon \mathcal{E}(u) < \infty\}.$$

Definition 1.5. We also assume the existence of a *self-similar measure* μ

$$\mu(A) = \sum_{j=1}^{J} \mu_j \mu(F_j^{-1}(A)), \tag{1.5}$$

with weights μ_j satisfying $0 < \mu_j < 1$ and $\sum_j \mu_j = 1$, and normalized so that $\mu(X) = 1$. With the notation of Definition 1.2, the measure of the m-cell K_ω is denoted by $\mu(K_\omega) = \mu_\omega := \mu_{\omega_1}\mu_{\omega_2}\ldots\mu_{\omega_m}$. The *standard measure* refers to the case $\mu_j = \frac{1}{J}$, for each j.

Remark 1.6. The renormalization factor r_j should be confused neither with the contraction factors $Lip(F_j)$ of the maps of the IFS, nor the weights μ_j of the self-similar measure μ. The values of these constants are completely independent.

Also, it should be noted that the existence of a self-similar energy asserted in Definition 1.4 is a strong assumption. While the the self-similar measures of Definition 1.5 always exist [Hut81], the existence of the self-similar energy is a much more delicate question; cf. [Sab97].

Definition 1.7. The *Laplacian* is defined weakly in terms of the energy form. For $u \in \operatorname{dom}\mathcal{E}$ and f continuous, one says $u \in \operatorname{dom}\Delta$ with $\Delta u = f$ iff

$$\mathcal{E}(u,v) = -\int_X f v \, d\mu, \quad \text{for all } v \in \operatorname{dom}_0 \mathcal{E}, \tag{1.6}$$

where $\operatorname{dom}_0 \mathcal{E}$ is the set of functions in $\operatorname{dom}\mathcal{E}$ which vanish on $\partial X = V_0$. Note that the Laplacian depends on the choice of measure μ.

More generally, if (1.6) holds with $f \in L^2(d\mu)$, then one says $u \in \operatorname{dom}_{L^2} \Delta$; and if

$$\mathcal{E}(u, v) = -\int_X v \, d\mu, \quad \text{for all } v \in \operatorname{dom}_0 \mathcal{E}, \tag{1.7}$$

for a finite signed measure μ with no atoms, then one says $u \in \operatorname{dom}_{\mathcal{M}} \Delta$.

It follows from (1.3), (1.5) and Definition 1.7 that Δ satisfies the scaling identity

$$\Delta(u \circ F_j) = r_j \mu_j (\Delta u) \circ F_j, \tag{1.8}$$

and pointwise formula given by the uniform limit

$$\Delta u(x) = \lim_{m\to\infty} \left(\int_X h_x^{(m)} \, d\mu \right)^{-1} \Delta_m u(x), \quad \text{for } x \in V_* \setminus V_0, \tag{1.9}$$

where $h_x^{(m)}$ is a piecewise harmonic spline satisfying $h_x^{(m)}(y) = \delta_{xy}$ for $y \in V_m$, and

$$\Delta_m u(x) = \sum_{y \underset{m}{\sim} x} c_{xy}(u(y) - u(x)), \quad \text{for } x \in V_m. \tag{1.10}$$

Roughly speaking, $h_x^{(m)}$ is a "tent" function with peak at x which vanishes outside the m-cell containing x. See [Str06, §2.1–§2.2] for details.

Definition 1.8. The *normal derivative* of a function u is computed at a boundary point $q \in V_0$ by

$$\partial_{\mathrm{n}} u(q) := \lim_{m\to\infty} \frac{1}{r_i^m} \sum_{y \underset{m}{\sim} q} (u(q) - u(y)), \qquad q \in V_0. \tag{1.11}$$

At a general junction point $x = F_\omega q$, the normal derivative is computed with respect to a specific m-cell K_ω:

$$\partial_{\mathrm{n}}^{K_\omega} u(x) = \partial_{\mathrm{n}}^{K_\omega} u(F_\omega q) := \frac{1}{r_{\omega_1} \cdots r_{\omega_m}} \partial_{\mathrm{n}}(u \circ F_\omega)(q). \tag{1.12}$$

1.2. **Statement of main result.**

Theorem 1.9. *Assume that λ is not a Dirichlet eigenvalue of Δ, and neither is $r_\omega \mu_\omega \lambda$, for any $\omega \in W_*$. For the Laplacian on X with Dirichlet boundary conditions, the resolvent kernel $G^{(\lambda)}$ defined by (1.1) is given by the formula*

$$G^{(\lambda)}(x, y) = \sum_{\omega \in W_*} r_\omega \Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1} x, F_\omega^{-1} y), \tag{1.13}$$

$$\text{where } \Psi^{(\lambda)}(x, y) := \sum_{p,q \in V_1 \setminus V_0} G_{pq}^{(\lambda)} \psi_p^{(\lambda)}(x) \psi_q^{(\lambda)}(y). \tag{1.14}$$

where convention stipulates $\Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1} x, F_\omega^{-1} y) = 0$ for x, y not in $F_\omega X$. In formula (1.14), $\psi_p^{(\lambda)}$ is the solution to the resolvent equation at level 1, i.e.

$$\begin{cases} (\lambda \mathbb{I} - \Delta)\psi_p^{(\lambda)} = 0, & \text{on each } K_j = F_j(X), \\ \psi_p^{(\lambda)}(q) = \delta_{pq}, & \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1, \end{cases} \tag{1.15}$$

where δ_{pq} is the Kronecker delta. The coefficients $G_{pq}^{(\lambda)}$ in (1.14) arise as the entries of the inverse of the matrix B given by

$$B_{pq}^{(\lambda)} := \sum_{K_j \ni q} \partial_n^{K_j} \psi_p^{(\lambda)}(q), \qquad q \in F_j(V_0), \tag{1.16}$$

where the sum is taken over all 1-cells containing q.

This result appears with proof as Theorem 3.12; a similar formula for Neumann boundary conditions appears in Theorem 4.2.

Remark 1.10. In (1.16) and elsewhere, we use the notation $\sum_{K_j \ni q}$ to indicate a sum being taken over the set $\{j \colon q \in K_j = F_j(X)\}$.

The rationale for the definitions (1.13)–(1.16) is best explained by the following heuristic argument and by comparison to [Str06, Thm. 2.6.1]. One would like $\Psi^{(\lambda)}$ to be a weak solution to the resolvent equation on a 1-cell $C = F_i(X)$, except at the boundary where some Dirac masses may appear. However, this implies that $r_i \Psi^{(r_i \mu_i \lambda)}(F_i^{-1}x, F_i^{-1}y)$ will be a weak solution on the 2-cell $F_i(C)$, and in the limit (1.13) gives a solution on the entire fractal. Each term added to the partial sum of (1.13) corresponds to canceling the Dirac masses at the previous stage and introducing new ones at the next; these are wiped away in the limit.

For $\Psi^{(\lambda)}$ to be a weak solution at level 1, we mean that if $u \in \operatorname{dom}\Delta$ and u vanishes on $\partial X = V_0$, then

$$\int_X \Psi^{(\lambda)}(x,y)(\lambda\mathbb{I} - \Delta)u(y)\,d\mu(y) = \sum_{p \in V_1 \setminus V_0} \psi_p^{(\lambda)}(x)u(p).$$

With (1.14) as given above, integration by parts and linearity give

$$\begin{aligned}\int_X \Psi^{(\lambda)}(x,y)(\lambda\mathbb{I} - \Delta)u(y)\,d\mu(y) &= \int_X \left[(\lambda\mathbb{I} - \Delta_y)\Psi^{(\lambda)}(x,y)\right] u(y)\,d\mu(y) \\ &= \sum_{p,q \in V_1 \setminus V_0} G_{pq}^{(\lambda)} \psi_p^{(\lambda)}(x) \int_X \left[(\lambda\mathbb{I} - \Delta)\psi_q^{(\lambda)}(y)\right] u(y)\,d\mu(y),\end{aligned}$$

where we used the notation Δ_y to indicate that the operator Δ is applied with respect to the variable y.

Now by (1.15), $\psi_q^{(\lambda)}$ satisfies the resolvent equation on the interior of the 1-cells, but $-\Delta\psi_q^{(\lambda)}$ has Dirac masses at the boundary points with weights $B_{qs}^{(\lambda)} := \sum_{K_j \ni s} \partial_{\mathrm{n}}^{K_j} \psi_q^{(\lambda)}(s)$. In other words, we have $\Delta\psi_q^{(\lambda)} = \lambda\psi_q^{(\lambda)}$ except on $V_1 \setminus V_0$, so that $(\lambda\mathbb{I} - \Delta)\psi_q^{(\lambda)}(y) = \sum_{s \in V_1 \setminus V_0} B_{qs}^{(\lambda)} \delta_s(y)$, where δ_s is the Dirac mass at s. Therefore, the calculation above may be continued:

$$\begin{aligned}\int_X \Psi^{(\lambda)}(x,y)(\lambda\mathbb{I} - \Delta)u(y)\,d\mu(y) &= \sum_{p,q,s \in V_1 \setminus V_0} \psi_p^{(\lambda)}(x) G_{pq}^{(\lambda)} B_{qs}^{(\lambda)} \int_X \delta_s(y)u(y)\,d\mu(y) \\ &= \sum_{p \in V_1 \setminus V_0} \psi_p^{(\lambda)}(x)u(p).\end{aligned}$$

The foregoing computation is the origin and motivation for (1.14)–(1.16). A key technical point is the use of a linear combination u of vectors $\psi_q^{(\lambda)}$ for which $(\lambda\mathbb{I} - \Delta)u$ is a *single* (weighted) Dirac mass at p. From the calculation, it is clear that this hinges on the invertibility of B; this is the significance of Lemma 3.7.

As mentioned just above, once the solution is obtained on level 1, it may be transferred to a cell $F_\omega(X)$ by rescaling appropriately. However, this is not sufficient to allow us to compute $(\lambda\mathbb{I} - \Delta_y)G^{(\lambda)}(x,y)$; some finesse is required to ensure that these solutions match where these cells intersect, that is, on the boundary points $V_{m+1} \setminus V_0$. Some further work is needed; this is carried out in the technical lemmas of §3.

2. The resolvent kernel for the unit interval

The unit interval $I = [0,1]$ has a self-similar structure derived from the IFS consisting of $F_1(x) = \frac{x}{2}$ and $F_2(x) = \frac{x}{2} + \frac{1}{2}$. In this section, we exploit this perspective to derive

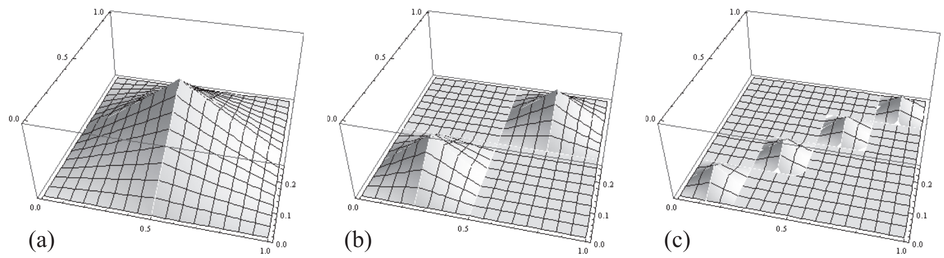

FIGURE 1. Mathematica plot of $\Psi^{(\lambda)}$ from Prop. 2.1 for $\lambda = 1$. (a) $\Psi^{(\lambda)}(x, y)$. (b) $\frac{1}{2}(\Psi^{(\lambda/4)}(2x, 2y) + \Psi^{(\lambda/4)}(2x-1, 2y-1))$. (c) $\frac{1}{4}(\Psi^{(\lambda/16)}(4x, 4y) + \Psi^{(\lambda/16)}(4x-1, 4y-1) + \Psi^{(\lambda/16)}(4x-2, 4y-2) + \Psi^{(\lambda/16)}(4x-3, 4y-3))$.

the resolvent kernel for the Dirichlet Laplacian on I by mimicking the construction of the Green function in [Kig01, §3.5] (see also [Str06, §2.6]). This exposition is intended to make the general case (presented in the next section) easier to digest. We build towards the result stated formally in Prop. 2.1.

Proposition 2.1. *Let $\Delta = \frac{d^2}{dx^2}$ be the Laplacian on the unit interval $I = [0, 1]$, taken with Dirichlet boundary conditions. If λ is not a Dirichlet eigenvalue of Δ, then the resolvent kernel $G^{(\lambda)}$ in (1.1) is given by*

$$G^{(\lambda)}(x, y) = \sum_{m=0}^{\infty} \sum_{|\omega|=m} \frac{1}{2^m} \Psi^{(\lambda/4^m)}(F_\omega^{-1}x, F_\omega^{-1}y), \tag{2.1}$$

$$\textit{for} \quad \Psi^{(\lambda)}(x, y) := \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda} \cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(y), \tag{2.2}$$

$$\textit{and} \quad \psi^{(\lambda)}(x) := \frac{1}{\sinh \frac{\sqrt{\lambda}}{2}} \begin{cases} \sinh \sqrt{\lambda}x, & x \le \frac{1}{2}, \\ \sinh \sqrt{\lambda}(1-x), & x \ge \frac{1}{2}, \end{cases} \tag{2.3}$$

where convention stipulates $\Psi^{(\lambda/4^m)}(F_\omega^{-1}x, F_\omega^{-1}y) = 0$ for x, y not in $F_\omega I$.

Remark 2.2 (A preview of the general case). Note that the sum in (2.1) is finite if $x \neq y$, or if $x = y$ is dyadic. More importantly, $\psi^{(\lambda)} = \psi^{(\lambda)}_{1/2}$ is the solution to the resolvent equation at level 1, i.e.

$$\begin{cases} (\lambda\mathbb{I} - \Delta)\psi^{(\lambda)} = 0, & \text{on } (0, \frac{1}{2}) \text{ and } (\frac{1}{2}, 1), \\ \psi^{(\lambda)}(0) = \psi^{(\lambda)}(1) = 0, & \text{and } \psi^{(\lambda)}(\frac{1}{2}) = 1. \end{cases} \tag{2.4}$$

In §3, we develop the resolvent kernel in the general case from these observations.

In keeping with the self-similar spirit of the sequel, we use the term *1-cell* in reference to the subintervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ in the following proof.

Proof of Prop. 2.1. On the unit interval I, one has the resolvent kernel

$$G^{(\lambda)}(x, y) = \frac{1}{\sqrt{\lambda} \sinh \sqrt{\lambda}} \begin{cases} \sinh \sqrt{\lambda}(1-y) \sinh \sqrt{\lambda}x & x \le y, \\ \sinh \sqrt{\lambda}y \sinh \sqrt{\lambda}(1-x), & x \ge y. \end{cases} \tag{2.5}$$

For $x \le \frac{1}{2} \le y$, one has

$$G^{(\lambda)}(x, y) = \frac{\sinh \sqrt{\lambda}(1-y) \sinh \sqrt{\lambda}x}{\sqrt{\lambda} \sinh \sqrt{\lambda}} \qquad \text{by (2.5)}$$

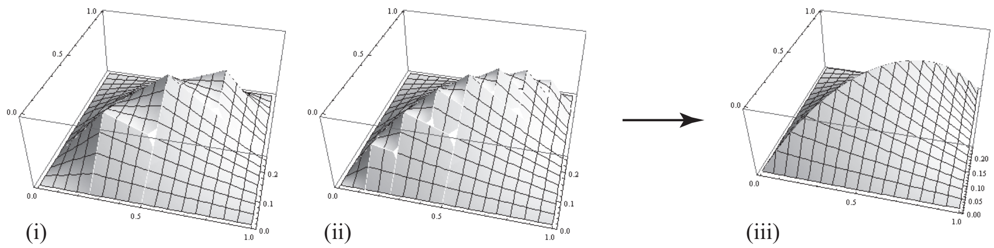

FIGURE 2. Mathematica plot of $G^{(\lambda)}$ for $\lambda = 1$ and two of its partial sums. (i) The sum of (a) and (b) in Fig. 1. (ii) The sum of (a), (b), (c) in Fig. 1. (iii) The resolvent kernel $G^{(\lambda)}(x, y)$ obtained in the limit.

$$
\begin{aligned}
&= \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} \cdot \frac{\sinh \sqrt{\lambda}x \sinh \sqrt{\lambda}(1-y)}{\sinh^2 \frac{\sqrt{\lambda}}{2}} \qquad && \sinh 2a = 2\sinh a \cosh a \\
&= \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(y) && \text{by (2.3).} \qquad (2.6)
\end{aligned}
$$

The same computation can be repeated for $y \le \frac{1}{2} \le x$ and hence (2.6) holds whenever x and y are in different 1-cells of I.

It remains to consider the case when both x and y lie in the same 1-cell of I. Suppose that $x \le y \le \frac{1}{2}$ and consider the difference

$$
\begin{aligned}
R(x, y) &:= G^{(\lambda)}(x, y) - \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(y) \\
&= \frac{\sinh \sqrt{\lambda}x (\sinh \sqrt{\lambda}(1-y) - \sinh \sqrt{\lambda}y)}{\sqrt{\lambda}\sinh \sqrt{\lambda}} \qquad (2.7) \\
&= \frac{\sinh \sqrt{\lambda}x \sinh \sqrt{\lambda}(\frac{1}{2} - y)}{\sqrt{\lambda}\sinh \frac{\sqrt{\lambda}}{2}} \\
&= \tfrac{1}{2} G^{(\lambda/4)}(2x, 2y), \qquad (2.8)
\end{aligned}
$$

where (2.7) follows by (2.5) and the identity $\sinh(1-a) - \sinh a = 2\sinh(\frac{1}{2} - a)\cosh \frac{1}{2}$. In the case when $y \le x \le \frac{1}{2}$, one also obtains $R(x, y) = \frac{1}{2}G^{(\lambda/4)}(2x, 2y)$. On the other hand, when x and y are both in the other 1-cell, one obtains (by analogous computations) that $R(x, y) = \frac{1}{2}G^{(\lambda/4)}(2x - 1, 2y - 1)$. Note that if λ is not a Dirichlet eigenvalue of Δ, then neither is $\lambda/4^m$ for any $m = 0, 1, 2, \ldots$. Consequently, if we define $\Psi^{(\lambda)}(x, y)$ as in (2.2), then formula (2.1) for $G^{(\lambda)}(x, y)$ follows. $\square$

Remark 2.3. It is interesting to note that the coefficient which appears in (2.6) is

$$
\frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} = \frac{1}{\psi^{(\lambda)\prime}(\frac{1}{2}-) - \psi^{(\lambda)\prime}(\frac{1}{2}+)}.
$$

Formally, this indicates $(\lambda\mathbb{I} - \Delta)G^{(\lambda)}(x, y) = \delta(x - y)$; compare to [Str06, (2.6.3)]. Also, observe that

$$
G^{(\lambda)}(x, \tfrac{1}{2}) = \frac{\sinh \frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\cosh \frac{\sqrt{\lambda}}{2}} \psi^{(\lambda)}(x)\psi^{(\lambda)}(\tfrac{1}{2}).
$$

At each successive iteration of (2.8), one is essentially "correcting" the formula on the diagonal for the m-cell with rescaled copies of the formula for the $(m+1)$-cell; Figures 1 and 2 are intended to explain this. In the next section, we follow this strategy for the construction of the resolvent kernel in the general case.

Remark 2.4. The procedure in the proof of Proposition 2.1 may also be carried out for the Neumann case: define a function $\varphi^{(\lambda)}$ to be the solution of

$$\begin{cases} (\lambda\mathbb{I}-\Delta)\varphi^{(\lambda)} = 0, & \text{on } [0,\tfrac12] \text{ and } [\tfrac12,1] \\ \frac{d}{dx}\varphi^{(\lambda)}(x) = 0, & x = 0,1 \\ \varphi^{(\lambda)}(\tfrac12) = 1, \end{cases}$$

which is given by

$$\varphi^{(\lambda)}(x) = \frac{1}{\cosh\frac{\sqrt{\lambda}}{2}} \begin{cases} \cosh\sqrt{\lambda}x, & x \le \tfrac12, \\ \cosh\sqrt{\lambda}(1-x), & x \ge \tfrac12. \end{cases}$$

Observe that in parallel to Remark 2.3, one again has

$$G_N^{(\lambda)}\left(x,\tfrac12\right) = \frac{\cosh\frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\sinh\frac{\sqrt{\lambda}}{2}}\,\varphi^{(\lambda)}(x)\,\varphi^{(\lambda)}\!\left(\tfrac12\right)$$

and

$$\frac{\cosh\frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\sinh\frac{\sqrt{\lambda}}{2}} = \frac{1}{\frac{d}{dx}\varphi^{(\lambda)}(\frac12-) - \frac{d}{dx}\varphi^{(\lambda)}(\frac12+)}.$$

By analogous computations, if we define

$$\Phi_N^{(\lambda)}(x,y) = \frac{\cosh\frac{\sqrt{\lambda}}{2}}{2\sqrt{\lambda}\sinh\frac{\sqrt{\lambda}}{2}}\,\varphi^{(\lambda)}(x)\,\varphi^{(\lambda)}(y),$$

then we obtain the Neumann resolvent kernel

$$G_N^{(\lambda)}(x,y) = \sum_{m=0}^{\infty}\sum_{|\omega|=m}\frac{1}{2^m}\Phi^{(\lambda/4^m)}(F_\omega^{-1}x, F_\omega^{-1}y).$$

3. The Dirichlet resolvent kernel for p.c.f. self-similar fractals

In this section, we proceed through a sequence of lemmas which will allow us to prove Theorem 1.9, which is stated in full in Theorem 3.12. On a first reading, the reader may wish to read Theorem 3.12 first, and then work through the lemmas in reverse order. We take one hypothesis of Theorem 1.9 as a blanket assumption throughout this section:

Assumption 3.1. None of the numbers $\lambda_\omega = \mu_\omega r_\omega \lambda$, for $\omega \in W_*$, is a Dirichlet eigenvalue of the Laplacian.

We construct the resolvent kernel formula according to the following rough outline:

(1) We build a solution $\eta_p^{(\lambda)}$ to the eigenfunction equation which takes the value 1 at one boundary point of X and is 0 on the other boundary points.
(2) We show how $\psi_p^{(\lambda)}$ may be written in terms of rescaled copies of $\eta_p^{(\lambda)}$, i.e., we decompose the solution around a point $p \in V_1 \setminus V_0$ into solutions for each cell containing p.
(3) We use this construction to obtain a solution on the cells of level m.
(4) We show how the $(m+1)$-level solution contains Dirac masses on V_m which cancel with the Dirac masses of the m-level solution, so that the sum over m is telescoping and yields a global solution.

The first two steps are carried out in §3.1. In §3.2, we collect some properties of $B^{(\lambda)}_{pq} := \sum_{K_j \ni q} \partial_n^{K_j} \psi^{(\lambda)}_p(q)$, as introduced in (1.15)[1]. For each λ, we think of $B^{(\lambda)}_{pq}$ as the entries of a matrix in p and q. Under Assumption 3.1, we show $B^{(\lambda)}$ is symmetric, invertible, and that $\lim_{\lambda\to 0} B^{(\lambda)} = B^{(0)}$. Finally, the remaining two steps are carried out in §3.3.

Throughout this section, we will need to analyze the properties of a continuous function that satisfies the λ-eigenfunction equation on all 1-cells, but whose the Laplacian may fail to be in L^2. Our motivation is that the Laplacian of such a function has Dirac masses at points $p \in V_1 \setminus V_0$ with coefficients that can be computed from the normal derivatives. The following result is standard; see [Str06, §2.5], for example.

Proposition 3.2. *If u is continuous and $\Delta u = v_j$ on each 1-cell $K_j = F_j(X)$, then Δu exists as a measure, as in (1.7), and*

$$\Delta u(x) = \sum_{j=1}^{J} v_j \chi_{K_j}(x) - \sum_{q \in V_1 \setminus V_0} \delta_q(x) \sum_{K_j \ni q} \partial_n^{K_j} u(q) \tag{3.1}$$

where $\delta_q(x)$ is a Dirac mass and $\partial_n^{K_j} u(q)$ is the normal derivative of u at q with respect to the cell K_j, and the sum is expressed in the notation of Remark 1.10.

3.1. **The basic building blocks of the resolvent kernel.**

Lemma 3.3. *For any λ that is not a Dirichlet eigenvalue of the Laplacian, and for each $p \in V_0$, there is a function $\eta^{(\lambda)}_p(x) \in \mathrm{dom}_{\mathcal{M}} \Delta$, as in (1.7), which solves*

$$\begin{cases} (\lambda\mathbb{I} - \Delta)\eta^{(\lambda)}_p(x) = 0, & \text{on } X, \\ \eta^{(\lambda)}_p(q) = \delta_{pq}, & \forall q \in V_0, \end{cases} \tag{3.2}$$

where δ_{pq} is the Kronecker delta. Moreover, if ζ_p is the harmonic function on X with $\zeta_p(q) = \delta_{pq}$, then

$$\eta^{(\lambda)}_p = \zeta_p - \lambda\theta^{(\lambda)}_p \qquad \text{on all of } X, \text{ and} \tag{3.3}$$

$$\partial_n \eta^{(\lambda)}_p(q) = \partial_n \zeta_p(q) - \lambda\kappa^{(\lambda)}_{pq} \qquad \text{for } q \in V_0, \tag{3.4}$$

where $\theta^{(\lambda)}_p$ and $\kappa^{(\lambda)}_{pq}$ are meromorphic functions of λ with poles at the Dirichlet eigenvalues of the Laplacian, and $\kappa^{(\lambda)}_{pq} = \kappa^{(\lambda)}_{qp}$.

Proof. Let $\{f_n\}$ denote the Dirichlet eigenfunctions of the Laplacian, with the corresponding eigenvalues λ_n arranged so that $\lambda_{n+1} \geq \lambda_n$; equality occurs iff λ_n has multiplicity greater than one. The functions f_n may be assumed orthonormal, and their span is dense in L^2. Consequently we may write $\zeta_p = \sum_n a_p(n) f_n$. The function

$$\theta^{(\lambda)}_p = \sum_n \frac{a_p(n)}{\lambda - \lambda_n} f_n$$

then satisfies

$$(\lambda\mathbb{I} - \Delta)\theta^{(\lambda)}_p = \sum_n \frac{a_p(n)}{\lambda - \lambda_n}(\lambda - \lambda_n) f_n = \sum_n a_p(n) f_n = \zeta_p$$

in the L^2 sense. We also see that $\Delta\theta^{(\lambda)}_p = \sum_n \lambda_n a_p(n) f_n/(\lambda - \lambda_n)$ is L^2 convergent, so $\theta^{(\lambda)}_p \in \mathrm{dom}_{L^2} \Delta$, as in (1.7). In particular, $\theta^{(\lambda)}_p$ is continuous and equal to zero on V_0.

Define $\eta^{(\lambda)}_p := \zeta_p - \lambda\theta^{(\lambda)}_p$. Then $(\lambda\mathbb{I} - \Delta)\eta^{(\lambda)}_p = (\lambda\mathbb{I} - \Delta)\zeta_p - \lambda\zeta_p = 0$, and for $q \in V_0$,

$$\eta^{(\lambda)}_p(q) = \zeta_p(q) = \delta_{pq}. \tag{3.5}$$

[1] Recall that $\psi^{(\lambda)}_p$ is the solution to the resolvent equation on level 1 as defined in (1.16).

To verify (3.4), we will need the fact that

$$\partial_{\mathrm{n}}\eta_p^{(\lambda)}(q) = \partial_{\mathrm{n}}\eta_q^{(\lambda)}(p), \tag{3.6}$$

which follows by computing the normal derivatives as follows:

$$\begin{aligned}
\partial_{\mathrm{n}}\eta_q^{(\lambda)}(p) - \partial_{\mathrm{n}}\eta_p^{(\lambda)}(q) &= \sum_{s\in V_0}\left(\eta_p^{(\lambda)}(s)\partial_{\mathrm{n}}\eta_q^{(\lambda)}(s) - \eta_q^{(\lambda)}(s)\partial_{\mathrm{n}}\eta_p^{(\lambda)}(s)\right) && \text{by (3.5)}\\
&= \int_X \left(\eta_p^{(\lambda)}(x)\Delta\eta_q^{(\lambda)}(x) - \eta_q^{(\lambda)}(x)\Delta\eta_p^{(\lambda)}(x)\right) d\mu(x) && \text{Gauss-Green}\\
&= 0, && \Delta\eta_s^{(\lambda)} = \lambda\eta_s^{(\lambda)}.
\end{aligned}$$

Now (3.4) follows via

$$\begin{aligned}
\partial_{\mathrm{n}}\zeta_p(q) - \partial_{\mathrm{n}}\eta_p^{(\lambda)}(q) &= \partial_{\mathrm{n}}\zeta_p(q) - \partial_{\mathrm{n}}\eta_q^{(\lambda)}(p) && \text{by (3.6)}\\
&= \sum_{s\in V_0}\left(\eta_q^{(\lambda)}(s)\partial_{\mathrm{n}}\zeta_p(s) - \zeta_p(s)\partial_{\mathrm{n}}\eta_q^{(\lambda)}(s)\right) && \text{by (3.5)}\\
&= \int_X \left(\eta_q^{(\lambda)}(x)\Delta\zeta_p(x) - \zeta_p(x)\Delta\eta_q^{(\lambda)}(x)\right) d\mu(x) && \text{Gauss-Green}\\
&= -\lambda\int_X \zeta_p(x)\eta_q^{(\lambda)}(x)\, d\mu(x) && \Delta\zeta_p = 0\\
&= \lambda\int_X \left(\lambda\zeta_p(x)\theta_q^{(\lambda)}(x) - \zeta_p(x)\zeta_q(x)\right) d\mu(x) && \text{by (3.3)}\\
&= \lambda^2\sum_n \frac{a_p(n)a_q(n)}{\lambda-\lambda_n} - \lambda\sum_n a_p(n)a_q(n)\\
&= \lambda\sum_n \frac{\lambda_n a_p(n)a_q(n)}{\lambda-\lambda_n}.
\end{aligned}$$

Define for each $p, q \in V_0$ the functions

$$\kappa_{pq}^{(\lambda)} := \sum_n \frac{\lambda_n a_p(n)a_q(n)}{\lambda-\lambda_n} \tag{3.7}$$

so that $\partial_{\mathrm{n}}\zeta_q(p) - \partial_{\mathrm{n}}\eta_q^{(\lambda)}(p) = \lambda\kappa_{pq}^{(\lambda)}$. It is evident that $\kappa_{pq}^{(\lambda)}$ is symmetric in p and q. It is also meromorphic in λ with poles at the points λ_n, as may be verified by writing the expansion on the disc of radius r centered at z (where $z \neq \lambda_n$ for any n, and $r = \inf_n |z-\lambda_n|/2$) as follows:

$$\kappa_{pq}^{(\lambda)} = \sum_n \frac{\lambda_n a_p(n)a_q(n)}{z-\lambda_n}\sum_{k=0}^{\infty}\left(\frac{z-\lambda}{z-\lambda_n}\right)^k = -\sum_{k=0}^{\infty}(\lambda-z)^k\sum_n \frac{\lambda_n a_p(n)a_q(n)}{(\lambda_n-z)^{k+1}}$$

and using the fact that $\{a_p(n)\}$ and $\{a_q(n)\}$ are in ℓ^2 hence their product is in ℓ^1, while $\lambda_n/(\lambda_n-z)^{k+1}$ is bounded for each k. Note that $r > 0$ because the eigenvalues of a p.c.f. fractal have no finite accumulation point; cf. [Kig01, §4.1]. An almost identical argument shows that $\theta_p^{(\lambda)}$ is meromorphic in λ with values in $\operatorname{dom}_{L^2}\Delta$, so the proof is complete. □

Corollary 3.4. *Let $p \in V_1 \setminus V_0$. If $r_j\mu_j\lambda$ is not a Dirichlet eigenvalue for any j with $p \in K_j$, then ψ and η are related via*

$$\psi_p^{(\lambda)}(x) = \begin{cases} \eta_{F_j^{-1}p}^{(r_j\mu_j\lambda)}(F_j^{-1}x) & \text{if } p, x \in K_j,\\ 0 & \text{otherwise.}\end{cases} \tag{3.8}$$

Proof. From (1.8) we have $\Delta(u \circ F_j^{-1}) = (r_j\mu_j)^{-1}(\Delta u) \circ F_j^{-1}$, for any u. Then from (1.15) and (3.2), one can observe that

$$\begin{cases} (\lambda\mathbb{I} - \Delta)\eta_p^{(r_j\mu_j\lambda)} \circ F_j^{-1} = 0, & \text{on } K_j = F_j(X) \\ \eta_p^{(r_j\mu_j\lambda)} \circ F_j^{-1}(q) = \delta_{F_j(p)q} & \forall q \in F_j(V_0). \end{cases} \qquad \square$$

Remark 3.5. It is helpful to compare (3.8) to the discussion of the unit interval, where (2.8) may be rewritten as

$$R(x,y) = \begin{cases} \frac{1}{2}G^{(\lambda/4)}(2x, 2y) & \text{if } x, y \in K_j, \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **The matrix $B^{(\lambda)}$.** In the construction of the resolvent kernel, the matrix $B^{(\lambda)}$ plays the same role as the transition matrix for the discrete Laplacian on V_1 in the corresponding argument of Kigami for the construction of the Dirichlet Green's function. We now collect some important properties of $B^{(\lambda)}$ for use below.

Lemma 3.6. *The matrix $B^{(\lambda)}$ is symmetric for any λ, and $\lim_{\lambda\to 0} B^{(\lambda)} = B^{(0)}$.*

Proof. From (3.6) we have $\partial_n^{K_j}\psi_p^{(\lambda)}(q) = \partial_n^{K_j}\psi_q^{(\lambda)}(p)$, and thus $B_{pq}^{(\lambda)} = B_{qp}^{(\lambda)}$. Then from (3.8), if $j_1, \dots j_k$ are those j for which K_j contains both p and q, then

$$\begin{aligned} B_{pq}^{(\lambda)} &= \sum_{i=1}^{k} \partial_n^{K_{j_i}} \left(\eta_{F_{j_i}^{-1}(p)}^{(r_{j_i}\mu_{j_i}\lambda)} \circ F_{j_i}^{-1} \right)(q) \\ &= \sum_{i=1}^{k} r_{j_i}^{-1} \partial_n^{K_{j_i}} \eta_{F_{j_i}^{-1}(p)}^{(r_{j_i}\mu_{j_i}\lambda)} \left(F_{j_i}^{-1}(q)\right) \\ &= \sum_{i=1}^{k} r_{j_i}^{-1} \partial_n^{K_{j_i}} \zeta_{F_{j_i}^{-1}(p)} \left(F_{j_i}^{-1}(q)\right) + \sum_{i=1}^{k} r_{j_i}^{-1} r_{j_i}\mu_{j_i}\lambda\, \kappa_{F_{j_i}^{-1}(p)F_{j_i}^{-1}(q)}^{(r_{j_i}\mu_{j_i}\lambda)} \qquad \text{by (3.4)} \\ &= B_{pq}^{(0)} + \lambda \sum_{i=1}^{k} \mu_{j_i} \kappa_{F_{j_i}^{-1}(p)F_{j_i}^{-1}(q)}^{(r_{j_i}\mu_{j_i}\lambda)} \end{aligned} \tag{3.9}$$

in which the final sum term is a meromorphic function of λ with poles at those λ for which $r_{j_i}\mu_{j_i}\lambda$ is a Dirichlet eigenvalue. We used the observation that the harmonic case with functions ζ is just the case $\lambda = 0$. From (3.9) it is also clear that $B_{pq}^{(\lambda)} \to B_{pq}^{(0)}$ as $\lambda \to 0$. $\square$

As noted in the discussion following the statement of Theorem 1.9, it is important that the action of $B^{(\lambda)}$ on the subspace $V_1 \setminus V_0$ is invertible.

Lemma 3.7. *If λ is not a Dirichlet eigenvalue then $B^{(\lambda)}$ is invertible.*

Proof. Suppose that $B^{(\lambda)} = \left[B_{pq}^{(\lambda)}\right]_{p,q\in V_1\setminus V_0}$ is not invertible, so there are values a_q (not all 0) for which $\sum_{q\in V_1\setminus V_0} B_{pq}^{(\lambda)} a_q = 0$. Define

$$u(x) := \sum_{q\in V_1\setminus V_0} a_q \psi_q^{(\lambda)}(x).$$

It is clear that $(\lambda\mathbb{I} - \Delta)u = 0$ on each 1-cell, and that $u\big|_{V_0} = 0$. Now using the notation from Remark 1.10, we compute the sum of the normal derivatives of u over cells containing p, for any $p \in V_1 \setminus V_0$:

$$\sum_{K_j \ni p} \partial_n^{K_j} u(p) = \sum_{q\in V_1\setminus V_0} a_q \sum_{K_j\ni p} \partial_n^{K_j} \psi_q^{(\lambda)}(p)$$

$$= \sum_{q \in V_1 \setminus V_0} a_q B^{(\lambda)}_{qp}$$
$$= 0,$$

where the last equality follows by applying the symmetry established in Lemma 3.6 to the initial assumption. So Proposition 3.2 implies Δu is continuous. It follows that $(\lambda \mathbb{I} - \Delta)u = 0$ on X, so u is a Dirichlet eigenfunction with eigenvalue λ, which is a contradiction. □

The next result is used to prove Lemma 3.11 and also makes use of (3.8).

Lemma 3.8. *For $p \in V_1 \setminus V_0$ and $q \in V_0$ we have*

$$\sum_{s \in V_1 \setminus V_0} B^{(\lambda)}_{ps} \eta^{(\lambda)}_q(s) = -B^{(\lambda)}_{pq} \tag{3.10}$$

Proof. For a 1-cell $K_j = F_j(X)$, the Gauss-Green formula gives

$$\sum_{s \in F_j(V_0)} \left(\psi^{(\lambda)}_p(s) \partial^{K_j}_{\mathrm{n}} \eta^{(\lambda)}_q(s) - \eta^{(\lambda)}_q(s) \partial^{K_j}_{\mathrm{n}} \psi^{(\lambda)}_p(s) \right)$$
$$= \int_{K_j} \left(\psi^{(\lambda)}_p(x) \Delta \eta^{(\lambda)}_q(x) - \eta^{(\lambda)}_q(x) \Delta \psi^{(\lambda)}_p(x) \right) d\mu(x) = 0$$

because both $\psi^{(\lambda)}_p(x)$ and $\eta^{(\lambda)}_q(x)$ are Laplacian eigenfunctions with eigenvalue λ on each 1-cell K_j. However for $s \in V_1$ we have $\psi^{(\lambda)}_p(s) = \delta_{ps}$, so this becomes

$$\partial^{K_j}_{\mathrm{n}} \eta^{(\lambda)}_q(p) = \sum_{s \in F_j(V_0)} \eta^{(\lambda)}_q(s) \partial^{K_j}_{\mathrm{n}} \psi^{(\lambda)}_p(s). \tag{3.11}$$

The continuity of the Laplacian of $\eta^{(\lambda)}_q$ at $p \in V_1 \setminus V_0$ implies that its normal derivatives sum to zero, as indicated by Proposition 3.2. Thus, summing over 1-cells yields

$$\begin{aligned}
0 = \sum_{j=1}^{J} \partial^{K_j}_{\mathrm{n}} \eta^{(\lambda)}_q(p) &= \sum_{j=1}^{J} \sum_{s \in F_j(V_0)} \eta^{(\lambda)}_q(s) \partial^{K_j}_{\mathrm{n}} \psi^{(\lambda)}_p(s) && \text{by (3.11)} \\
&= \sum_{s \in V_1} \eta^{(\lambda)}_q(s) \sum_{K_j \ni s} \partial^{K_j}_{\mathrm{n}} \psi^{(\lambda)}_p(s) && \text{interchange} \\
&= \sum_{s \in V_0} \eta^{(\lambda)}_q(s) B^{(\lambda)}_{ps} + \sum_{s \in V_1 \setminus V_0} \eta^{(\lambda)}_q(s) B^{(\lambda)}_{ps} && \text{split} \\
&= B^{(\lambda)}_{pq} + \sum_{s \in V_1 \setminus V_0} B^{(\lambda)}_{ps} \eta^{(\lambda)}_q(s) && \eta^{(\lambda)}_q(s) = \delta_{qs} \text{ on } V_0
\end{aligned}$$

where we used the sum notation of Remark 1.10. □

3.3. **Construction of the resolvent kernel.** Now that we have obtained some necessary properties of $B^{(\lambda)}$, we can proceed with the development of a sequence of technical lemmas required for the proof of the main result. We begin with another corollary of Proposition 3.2.

Corollary 3.9. *If $p \in V_1$ and λ satisfies Assumption 3.1, then*

$$(\lambda \mathbb{I} - \Delta) \psi^{(\lambda)}_p = \sum_{q \in V_1 \setminus V_0} B^{(\lambda)}_{pq} \delta_q. \tag{3.12}$$

Proof. With $\psi^{(\lambda)}_p$ and $B^{(\lambda)}_{pq}$ defined as in (1.15)–(1.16), this is clear from (3.1). □

Remark 3.10. From the definition in (1.16), we have $B^{(\lambda)}_{pq} = \sum_{K_j \ni q} \partial_{\mathrm{n}}^{K_j} \psi^{(\lambda)}_p(q)$ for $q \in F_j(V_0)$. Thus Corollary 3.9 expresses the fact that an application of the resolvent to $\psi^{(\lambda)}_p$ leaves behind nothing but a Dirac mass at every point of $V_1 \setminus V_0$, each weighted by the sum of the normal derivatives of $\psi^{(\lambda)}_p$.

The conclusion of the following lemma appears very technical but it expresses a straightforward idea: at each stage m, our formula for the resolvent corrects Dirac masses at the m^{th} level and introduces new ones at the $(m+1)^{\text{th}}$. Thus, summing over m (as we do in Theorem 3.12) produces a telescoping series. This makes precise the comment "these are wiped away in the limit" from the introductory discussion of the main result.

Lemma 3.11. *Define $\xi^{(\lambda)}_{p,m}$ to be the unique function solving*

$$\begin{cases} (\Delta - \lambda)\xi^{(\lambda)}_{p,m} = 0, & \text{on all } m\text{-cells}, \\ \xi^{(\lambda)}_{p,m}(q) = \delta_{pq}, & \text{for } p \in V_m \setminus V_0 \text{ and } q \in V_m. \end{cases} \tag{3.13}$$

Then one has the identity

$$(\lambda \mathbb{I} - \Delta_y) \sum_{|\omega|=m} r_\omega \Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1}x, F_\omega^{-1}y) = \sum_{p \in V_{m+1} \setminus V_0} \xi^{(\lambda)}_{p,m+1}(x)\delta_p(y) - \sum_{q \in V_m \setminus V_0} \xi^{(\lambda)}_{q,m}(x)\delta_q(y).$$

Proof. Since $\Psi^{(r_\omega \mu_\omega \lambda)}$ is a sum of functions satisfying the λ-eigenfunction equation on the level 1 cells K_j, it is immediate that

$$(\lambda - \Delta_y)\Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1}x, F_\omega^{-1}y) = 0, \qquad \text{for } y \notin V_{m+1}.$$

By Proposition 3.2, we therefore need only compute the sum of normal derivatives at points of V_{m+1}.

(1) First suppose that $z \in V_{m+1} \setminus V_m$ with $z = F_\omega p$ for some $|\omega| = m$ and $p \in V_1 \setminus V_0$, so that

$$\Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1}x, F_\omega^{-1}z) = \sum_{s,t \in V_1 \setminus V_0} G^{(r_\omega \mu_\omega \lambda)}_{st}\, \psi^{(r_\omega \mu_\omega \lambda)}_s(F_\omega^{-1}x)\, \psi^{(r_\omega \mu_\omega \lambda)}_t(F_\omega^{-1}z),$$

and collecting normal derivatives at z yields

$$\begin{aligned} \sum_{F_\omega(K_j) \ni z} \partial_{\mathrm{n}}^{F_\omega(K_j)} \Big(\Psi^{(r_\omega \mu_\omega \lambda)}(F_\omega^{-1}x, F_\omega^{-1}z) \Big) &= \sum_{s,t \in V_1 \setminus V_0} G^{(r_\omega \mu_\omega \lambda)}_{st} \psi^{(r_\omega \mu_\omega \lambda)}_s(F_\omega^{-1}x) \sum_{F_\omega(K_j) \ni z} \partial_{\mathrm{n}}^{F_\omega(K_j)} \psi^{(r_\omega \mu_\omega \lambda)}_t(F_\omega^{-1}z) \\ &= r_\omega^{-1} \sum_{s,t \in V_1 \setminus V_0} G^{(r_\omega \mu_\omega \lambda)}_{st}\, \psi^{(r_\omega \mu_\omega \lambda)}_s(F_\omega^{-1}x)\, B^{(r_\omega \mu_\omega \lambda)}_{tp}, \end{aligned} \tag{3.14}$$

because

$$\begin{aligned} B^{(r_\omega \mu_\omega \lambda)}_{tp} &= \sum_{K_j \ni p} \partial_{\mathrm{n}}^{K_j} \psi^{(r_\omega \mu_\omega \lambda)}_t(p) && \text{by (1.16)} \\ &= \sum_{K_j \ni F_\omega^{-1} z} \partial_{\mathrm{n}}^{K_j} \psi^{(r_\omega \mu_\omega \lambda)}_t(F_\omega^{-1}z) && p = F_\omega^{-1}z \in K_j \\ &= r_\omega \sum_{F_\omega(K_j) \ni z} \partial_{\mathrm{n}}^{F_\omega(K_j)} \left(\psi^{(r_\omega \mu_\omega \lambda)}_t \circ F_\omega^{-1} \right)(z), \end{aligned}$$

where the last line follows from $\partial_{\mathrm{n}}^{K_\omega} u(F_\omega^{-1} q_i) = r_\omega \partial_{\mathrm{n}} (u \circ F_\omega^{-1})(q_i)$; cf. (1.12).

Continuing the computation from (3.14) and making use of $G := B^{-1}$, we have

$$\begin{aligned}\sum_{F_\omega(K_j)\ni z} \partial_{\mathrm{n}}^{F_\omega(K_j)}\Big(\Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}z)\Big) &= r_\omega^{-1} \sum_{s\in V_1\setminus V_0} \delta_{sp}\left(\psi_s^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}(x))\right)\\ &= r_\omega^{-1}\psi_p^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}(x))\\ &= r_\omega^{-1}\xi_{z,m+1}^{(\lambda)}(x)\end{aligned}$$

thus showing that $(\lambda\mathbb{I} - \Delta_y)$ has a Dirac mass $\xi_{z,m+1}^{(\lambda)}(x)\delta_z(y)$ at $z \in V_{m+1} \setminus V_m$.

(2) Next consider a point $z \in V_m \setminus V_0$. In this case there are several words ω_i for which $z = F_{\omega_i}(p_i)$ for some $p_i \in V_0$. For such a word ω and such a p we substitute from Lemma 3.8 into (3.14), obtaining

$$\begin{aligned}\sum_{K_j\ni p} \partial_{\mathrm{n}}^{F_\omega(K_j)}\Big(r_\omega\Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}z)\Big) &= -\sum_{q,s,t\in V_1\setminus V_0} G_{s,t}^{(r_\omega\mu_\omega\lambda)}\,\psi_s^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x)\,B_{tq}^{(r_\omega\mu_\omega\lambda)}\eta_p^{(r_\omega\mu_\omega\lambda)}(q)\\ &= -\sum_{q,s\in V_1\setminus V_0} \delta_{sq}\,\psi_s^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x)\,\eta_p^{(r_\omega\mu_\omega\lambda)}(q)\\ &= -\sum_{q\in V_1\setminus V_0} \psi_q^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x)\,\eta_p^{(r_\omega\mu_\omega\lambda)}(q). \qquad (3.15)\end{aligned}$$

The result is clearly a piecewise λ-eigenfunction on level $(m+1)$ with respect to the x variable, so is determined by its values on V_{m+1}. In each of the terms (3.15), the values are nonzero only at the points of V_{m+1} that neighbor z in $F_\omega(X)$, and they are easily seen to coincide with $\xi_{z,m+1}^{(\lambda)} - \xi_{z,m}^{(\lambda)}$ at these points. Summing over all cells, we conclude that at each $z \in V_m \setminus V_0$ the operator $(\lambda\mathbb{I} - \Delta)$ has a Dirac mass $\left(\xi_{z,m+1}^{(\lambda)} - \xi_{z,m}^{(\lambda)}\right)\delta_z(y)$, and the result follows. ◻

Theorem 3.12. *Let $\psi_p^{(\lambda)}$ be the solution to the resolvent equation at level 1, i.e.*

$$\begin{cases}(\lambda\mathbb{I} - \Delta)\psi_p^{(\lambda)} = 0, & \text{on each } K_j = F_j(X),\\ \psi_p^{(\lambda)}(q) = \delta_{pq}, & \text{for } p \in V_1 \setminus V_0 \text{ and } q \in V_1,\end{cases} \qquad (3.16)$$

where δ_{pq} is the Kronecker delta.

Define the kernel

$$G^{(\lambda)}(x,y) = \sum_{\omega\in W_*} r_\omega\Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}y), \qquad (3.17)$$

$$\text{where } \Psi^{(\lambda)}(x,y) := \sum_{p,q\in V_1\setminus V_0} G_{pq}^{(\lambda)}\psi_p^{(\lambda)}(x)\psi_q^{(\lambda)}(y). \qquad (3.18)$$

The coefficients $G_{pq}^{(\lambda)}$ in (3.18) are the entries of the inverse of the matrix B given by

$$B_{pq}^{(\lambda)} := \sum_{K_j\ni q} \partial_n^{K_j}\psi_p^{(\lambda)}(q), \qquad q \in F_j(V_0), \qquad (3.19)$$

the sum taken over all 1-cells containing q.

For λ satisfying Assumption 3.1, $G^{(\lambda)}(x,y)$ is symmetric and continuous in x and y, and is in $\mathrm{dom}_{\mathcal{M}}\,\Delta_y$ with $(\lambda\mathbb{I} - \Delta_y)G^{(\lambda)}(x,y) = \delta_x(y)$. As it vanishes on V_0, it is the Dirichlet resolvent of the Laplacian.

Proof. The symmetry of $G^{(\lambda)}(x,y)$ is obvious. Next, note that

$$(\lambda\mathbb{I} - \Delta_y)\sum_{m=0}^{M}\sum_{|\omega|=m} r_\omega\Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}y)$$

$$= \sum_{m=0}^{M} \left(\sum_{p \in V_{m+1} \setminus V_0} \xi^{(\lambda)}_{p,m+1}(x)\delta_p(y) - \sum_{p \in V_m \setminus V_0} \xi^{(\lambda)}_{p,m}(x)\delta_p(y) \right)$$
$$= \sum_{p \in V_{M+1} \setminus V_0} \xi^{(\lambda)}_{p,M+1}(x)\delta_p(y)$$

by Lemma 3.11, so that

$$\lim_{M\to\infty} (\lambda\mathbb{I} - \Delta_y) \sum_{m=0}^{M} \sum_{|\omega|=m} r_\omega \Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}y) = \delta_x(y),$$

in the sense of weak-$*$ convergence. It follows that $G^{(\lambda)}(x,y)$ is in $\mathrm{dom}_{\mathcal{M}}(\Delta_y)$ and that $(\lambda\mathbb{I} - \Delta_y)G^{(\lambda)}(x,y) = \delta_x(y)$.

All that remains is to see that $G^{(\lambda)}(x,y)$ is continuous. However, Lemma 3.6 shows $B^{(r_\omega\mu_\omega\lambda)}_{pq} \to B^{(0)}_{pq}$ as $|\omega| \to \infty$, and hence $G^{(r_\omega\mu_\omega\lambda)}_{pq} \to G^{(0)}_{pq}$. In a similar manner, the relation $\eta^{(r_\omega\mu_\omega\lambda)}_p = \zeta_p + r_\omega\mu_\omega\lambda\theta^{(r_\omega\mu_\omega\lambda)}_p$ from Lemma 3.3 shows that $\eta^{(r_\omega\mu_\omega\lambda)}_p \to \zeta_p$ as $|\omega| \to \infty$; in particular we find that $\psi^{(r_\omega\mu_\omega\lambda)}_p \to \psi^{(0)}_p$, and the latter is piecewise harmonic and bounded by 1. The conclusion is that $\Psi^{(r_\omega\mu_\omega\lambda)}$ is bounded as $|\omega| \to \infty$, and since r_ω is a product of $|\omega|$ terms, all of which are bounded by $\max_i r_i < 1$,

$$G^{(\lambda)}(x,y) = \sum_{m=0}^{\infty} \sum_{|\omega|=m} r_\omega \Psi^{(r_\omega\mu_\omega\lambda)}(F_\omega^{-1}x, F_\omega^{-1}y)$$

is bounded by a convergent geometric series. Note that, for each m, only a finite number of terms in the second sum are nonzero. As all terms are continuous, so is $G^{(\lambda)}$. □

4. The Neumann resolvent kernel for p.c.f. self-similar fractals

In Theorem 4.2, we give the formula for the Neumann resolvent kernel.

Lemma 4.1. *If λ is not a Neumann eigenvalue then there is $C^{(\lambda)}_{pq}$ such that*

$$\sum_{q \in V_0} C^{(\lambda)}_{pq} \partial_n \eta^{(\lambda)}_q(x) = \delta_{px}$$

for $x \in V_0$, and $C^{(\lambda)}_{pq}$ is symmetric in p and q.

Proof. Since λ is not a Neumann eigenvalue, the set of vectors $\left\{ \left(\partial_n \eta^{(\lambda)}_q(x) \right)_{x\in V_0} \right\}_{q\in V_0}$ is linearly independent, whence the existence of the $C^{(\lambda)}_{pq}$ is immediate. Symmetry follows from (3.6) because the matrix $\left[C^{(\lambda)}_{pq} \right]$ is the inverse of the symmetric matrix $\left[\partial_n \eta^{(\lambda)}_p(q) \right]$. □

From this and Theorem 3.12 we may readily deduce the following result.

Theorem 4.2. *If λ satisfies Assumption 3.1 and also is not a Neumann eigenvalue, then*

$$G^{(\lambda)}_N(x,y) = G^{(\lambda)}(x,y) + \sum_{p,q\in V_0} C^{(\lambda)}_{pq} \eta^{(\lambda)}_p(x) \eta^{(\lambda)}_q(y) \tag{4.1}$$

is symmetric, is in $\mathrm{dom}_{\mathcal{M}}(\Delta_y)$, and satisfies $(\lambda - \Delta_y)G^{(\lambda)}_N(x,y) = \delta_x(y)$ on $X \setminus V_0$. It has vanishing normal derivatives on V_0 and is therefore the Neumann resolvent kernel of the Laplacian.

Proof. The symmetry of $G_N^{(\lambda)}(x,y)$ is immediate from the symmetry of $G^{(\lambda)}(x,y)$ and of $C_{pq}^{(\lambda)}$. Both $G^{(\lambda)}(x,y)$ and $\eta_p^{(\lambda)}(y)$ are in $\mathrm{dom}_{\mathcal{M}}(\Delta_y)$ and $(\lambda-\Delta_y)\eta_p^{(\lambda)}(y) = 0$ on $X \setminus V_0$ so $(\lambda-\Delta_y)G_N^{(\lambda)}(x,y) = (\lambda-\Delta_y)G^{(\lambda)}(x,y) = \delta_x(y)$ on $X \setminus V_0$.

It remains to prove the assertion about the normal derivatives. We will use the notation $(\partial_{\mathrm{n}})_y G^{(\lambda)}$ for the normal derivative of $G^{(\lambda)}(x,y)$ with respect to its second variable. Since $G^{(\lambda)}(x,y) \in \mathrm{dom}_{\mathcal{M}}(\Delta_y)$ it has a normal derivative at $p \in V_0$, and by the Gauss-Green formula,

$$\begin{aligned}
(\partial_{\mathrm{n}})_y G^{(\lambda)}(x,p) &= \sum_{s\in V_0}\Big((\partial_{\mathrm{n}})_y G^{(\lambda)}(x,s)\eta_p^{(\lambda)}(s) - G^{(\lambda)}(x,s)\partial_n\eta_p^{(\lambda)}(s)\Big)\\
&= \int_X \Big(\big(\Delta_s G^{(\lambda)}(x,s)\big)\,\eta_p^{(\lambda)}(s) - G^{(\lambda)}(x,s)\left(\Delta_s\eta_p^{(\lambda)}(s)\right)\Big)\,d\mu(s)\\
&= \int_X (\Delta_s - \lambda)\,G^{(\lambda)}(x,s)\eta_p^{(\lambda)}(s)\,d\mu(s)\\
&= -\eta_p^{(\lambda)}(x) \qquad (4.2)
\end{aligned}$$

where at the first step we used that $G^{(\lambda)}(x,s) = 0$ for $s \in V_0$ and at the last step we used $(\Delta_s - \lambda)\,G^{(\lambda)}(x,s) = -\delta_x(s)$ as a measure. It follows that at each $p \in V_0$, the normal derivative of (4.1) vanishes:

$$\begin{aligned}
(\partial_{\mathrm{n}})_y G_N^{(\lambda)}(x,p) &= -\eta_p^{(\lambda)}(x) + (\partial_{\mathrm{n}})_y \sum_{q,s\in V_0} C_{qs}^{(\lambda)}\eta_q^{(\lambda)}(x)\eta_s^{(\lambda)}(p) && \text{by (4.2)}\\
&= -\eta_p^{(\lambda)}(x) + \sum_{q\in V_0}\delta_{qx}\eta_q^{(\lambda)}(x) && \text{by Lemma 4.1}\\
&= 0.
\end{aligned}$$
□

5. Example: the Sierpinski gasket SG

Recall the harmonic extension algorithm as described in [Str06, §1.3]: if the values of a function u are specified at the points of V_0 and written as a vector

$$u\big|_{V_0} = \begin{bmatrix} u(p_0)\\ u(p_1)\\ u(p_2)\end{bmatrix},$$

then the harmonic extension of u to $F_i(V_0)$ (the boundary points of the 1-cell $F_i(SG)$) is given by

$$u\big|_{F_iV_0} = A_i u\big|_{V_0} = \begin{bmatrix} u(F_ip_0)\\ u(F_ip_1)\\ u(F_ip_2)\end{bmatrix},$$

where

$$A_0 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0\\ 2 & 2 & 1\\ 2 & 1 & 2\end{bmatrix},\quad A_1 = \frac{1}{5}\begin{bmatrix} 2 & 2 & 1\\ 0 & 5 & 0\\ 1 & 2 & 2\end{bmatrix},\quad \text{and}\quad A_2 = \frac{1}{5}\begin{bmatrix} 2 & 1 & 2\\ 1 & 2 & 2\\ 0 & 0 & 5\end{bmatrix}$$

are the harmonic extension matrices. In general, $u\big|_{F_\omega V_0} = A_\omega u\big|_{V_0}$, where $A_\omega = A_{\omega_m}\cdots A_{\omega_1}$. Thus, the harmonic extension matrices allow one to construct a harmonic function with specified boundary values. Similarly, spectral decimation provides matrices which allow

one to construct an eigenfunction with specified boundary values. For example,

$$A_0(\lambda) = \frac{1}{(5-\lambda)(2-\lambda)} \begin{bmatrix} (5-\lambda)(2-\lambda) & 0 & 0 \\ (4-\lambda) & (4-\lambda) & 2 \\ (4-\lambda) & 2 & (4-\lambda) \end{bmatrix} \tag{5.1}$$

is the analogue of $A_0 = A_0(0)$. By the usual caveats of spectral decimation, these extension matrices can only be used when λ is not a (Dirichlet) eigenvalue.

Remark 5.1 (Spectral decimation). A very brief outline of the method of spectral decimation is as follows.

(1) Begin on some level $m = m_0$ with u_m and λ_m that satisfy $-\Delta_m u_m = \lambda_m u_m$ on $V_m \setminus V_0$.
(2) Extend u_m to a function u_m on $V_{m+1} \setminus V_0$ by comparing the eigenvalue equations from each level.
(3) Obtain a collection of extension matrices, one for each mapping in the original IFS, and a rational function ϱ which relates the eigenvalues on one level to the eigenvalues on the previous level by $\varrho(\lambda_m) = \lambda_{m-1}$.
(4) Inductively construct a sequence $\{\lambda_{m_o}, \lambda_{m_o+1}, \lambda_{m_o+2}, \dots\}$ by choosing λ_{m+1} from the set $\varrho^{-1}(\lambda_m)$ for each m.

For every such sequence that converges, $\alpha \lim_{m\to\infty} \beta^m \lambda_m$ will be an eigenvalue of Δ on X, where α and β are constants specific to X. For the Sierpinski Gasket SG, $\alpha = \frac{3}{2}$ and $\beta = 3\frac{5}{3} = 5$. Note that the calculations in (2)–(3) will forbid certain choices, so some care must be taken in the construction of $\{\lambda_m\}$. See [Str06, §3] for more details.

To obtain the numbers $B_{pq}^{(\lambda)}$ (appearing in (1.16)) for the Sierpinski Gasket SG, we find the normal derivatives of the eigenfunction that has boundary values $(1, 0, 0)$, as computed at each point of V_0. If $(\lambda\mathbb{I} - \Delta)u = 0$ but u is not a Dirichlet eigenfunction, then consider u on $F_0^m(V_0)$. By spectral decimation, this is given by

$$u\big|_{F_0^m(V_0)} = A_0(\lambda_m)\cdots A_0(\lambda_1)u\big|_{V_0}$$

where the matrix $A_0(\lambda)$ is as in (5.1). We actually only need the values of the *normal derivative*

$$\partial_n u(q_i) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \left(2u(q_i) - u(F_i^m q_{i-1}) - u(F_i^m q_{i+1})\right), \qquad q_i \in V_0. \tag{5.2}$$

The factor $\frac{5}{3}$ arises here because $r_j = \frac{3}{5}$ for each j in (1.3) for SG; see also (1.12). The calculation of $r_j = \frac{3}{5}$ is given in [Str06, §1.3].

It is extremely easy to compute the normal derivatives of a harmonic function: one does not need to compute the limit, as all terms of the sequence are equal; see [Str06, (2.3.9)]. Therefore, our approach is to obtain a harmonic function which coincides with u on $F_0^m(V_0)$. The limit of the normal derivatives of these harmonic functions will be the normal derivative of u. An alternative interpretation would be to interpret the harmonic functions on SG as the analogue of the linear functions on I. Consequently, the tangent to a point of SG should be given by a harmonic function plus a constant, provided the tangent exists. This is the motivating idea of [DRS09].

Multiplication by A_0^{-m} allows one to find the required harmonic function at stage m; rewriting the normal derivative (5.2) in vector notation, one has

$$\begin{aligned} \left(\frac{5}{3}\right)^m (2,-1,-1)\cdot u\big|_{F_0^m(V_0)} &= (2,-1,-1)\cdot A_0^{-m} u\big|_{F_0^m(V_0)} \\ &= (2,-1,-1)\cdot A_0^{-m} A_0(\lambda_m)\cdots A_0(\lambda_1)u\big|_{V_0}. \end{aligned}$$

It therefore suffices to understand the limit $\lim_m A_0^{-m} A_0(\lambda_m)\cdots A_0(\lambda_1)$; this was computed in [DRS09]. The following theorem is the main result of [DRS09], taken with $m_0 = 0$.

Theorem 5.2. *Let $\alpha = (0,1,1)^T$, $\beta = (0,1,-1)^T$, $\gamma_m = (4, 4-\lambda_m, 4-\lambda_m)^T$. If neither of the values 2 or 5 occur in the sequence λ_m, then*

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{0+k})\cdots A_0(\lambda_{0+1})\,\alpha = \frac{4\lambda}{3\cdot 5^0\lambda_0(2-\lambda_{0+1})}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_{0+j}}{3}\right)\alpha$$
$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{0+k})\cdots A_0(\lambda_{0+1})\,\beta = \frac{2\lambda}{3\cdot 5^0\lambda_0}\beta$$
$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{0+k})\cdots A_0(\lambda_{0+1})\,\gamma_0 = (4,4,4)^T$$

In particular, this can be used to get the desired normal derivative. We know that all we need do is compute

$$(2,-1,-1)\cdot\left(\lim_m A_0^{-m} A_0(\lambda_m)\cdots A_0(\lambda_1)\right)u\big|_{V_0}. \tag{5.3}$$

The boundary data $u\big|_{V_0}$ is be taken to be $(1,0,0)$ when computing the normal derivative at the point p where $u(p) = 1$, and $(0,1,0)$ at a point where $u(p) = 0$ (these two points are the same by symmetry). Writing

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{4}\begin{bmatrix}4\\4-\lambda_0\\4-\lambda_0\end{bmatrix} - \frac{4-\lambda_0}{4}\begin{bmatrix}0\\1\\1\end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\1\\-1\end{bmatrix},$$

we find that

$$\lim_m A_0^{-m} A_0(\lambda_m)\cdots A_0(\lambda_1)\begin{bmatrix}1\\0\\0\end{bmatrix}$$
$$= \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{4-\lambda_0}{4}\,\frac{4\lambda}{3\lambda_0(2-\lambda_1)}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_j}{3}\right)\begin{bmatrix}0\\1\\1\end{bmatrix}$$

and by (5.3), the normal derivative is

$$\frac{2(4-\lambda_0)\lambda}{3\lambda_0(2-\lambda_1)}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_j}{3}\right).$$

The normal derivative at the other point is computed by first finding

$$\lim_m A_0^{-m} A_0(\lambda_m)\cdots A_0(\lambda_1)\begin{bmatrix}0\\1\\0\end{bmatrix}$$
$$= \frac{1}{2}\,\frac{4\lambda}{3\lambda_0(2-\lambda_1)}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_j}{3}\right)\begin{bmatrix}0\\1\\1\end{bmatrix} + \frac{1}{2}\,\frac{2\lambda}{3\lambda_0}\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

and then taking the inner product with $(2,-1,-1)$, which cancels the second vector to leave

$$\frac{-4\lambda}{3\lambda_0(2-\lambda_1)}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_j}{3}\right).$$

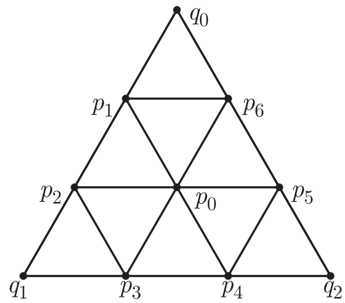

FIGURE 3. The 1-cells of SG_3.

It seems logical at this point to define a function

$$\tau(\lambda) = \frac{4\lambda}{3\lambda_0(2-\lambda_1)} \prod_{j=2}^{\infty} \left(1 - \frac{\lambda_j}{3}\right) \tag{5.4}$$

and to write the normal derivative at the point where the 1 occurs as $(4-\lambda_0)\tau(\lambda)/2$ and that at the point where the 0 occurs as $-\tau(\lambda)$. We note that for a non-Dirichlet eigenfunction, none of the values $2, 5, 6$ occur in λ_m for $m \geq 1$ so the term $(2 - \lambda_1)$ in the denominator cannot be zero. It follows that 3 does not occur for $m \geq 2$ and therefore that $\tau(\lambda) \neq 0$ in this case. An exception to our formula as currently written occurs when $\lambda = 0$, because then also $\lambda_0 = 0$, but the function $\tau(\lambda)$ is easily shown to have a continuous extension to $\lambda = 0$ with $\tau(0) = 1$; cf. [DRS09]. With this correction, our formula is also valid for the harmonic case.

It is now easy to write the entries of the matrix $B^{(\lambda)}_{pq}$ appearing in (1.16). The term $B^{(\lambda)}_{pp}$ has two copies of the normal derivative $(4 - \lambda_0)\tau(\lambda)/2$, and the term $B^{(\lambda)}_{pq}$ has a single copy of $-\tau(\lambda)$ at each $q \in V_1$ that is not equal to p. Both are on 1-cells rather than the whole of SG, so there is an extra factor $5/3$ in their normal derivatives. As a result, the matrix is

$$B = \frac{5}{3} \begin{bmatrix} (4-\lambda_0)\tau(\lambda) & -\tau(\lambda) & -\tau(\lambda) \\ -\tau(\lambda) & (4-\lambda_0)\tau(\lambda) & -\tau(\lambda) \\ -\tau(\lambda) & -\tau(\lambda) & (4-\lambda_0)\tau(\lambda) \end{bmatrix}$$

and we should invert this to get the matrix G_{pq} for the Green's function. Since

$$\det \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} = (a-b)^2(a+2b),$$

the matrix B is invertible iff $\lambda_0 \neq 2, 5$, in which case

$$G^{(\lambda)} = \frac{3}{5(5-\lambda_0)(2-\lambda_0)\tau(\lambda)} \begin{bmatrix} (3-\lambda_0) & 1 & 1 \\ 1 & (3-\lambda_0) & 1 \\ 1 & 1 & (3-\lambda_0) \end{bmatrix}.$$

Note that this is consistent with the harmonic case where $\lambda_0 = 0$ and $\tau(0) = 1$ gives factors $9/50$ for $G^{(\lambda)}_{pp}$ and $3/50$ for $G^{(\lambda)}_{pq}$ with $p \neq q$; see [Str06, (2.6.25)].

6. EXAMPLE: SG_3, A VARIANT OF THE SIERPINSKI GASKET

6.1. **The Laplacian on SG_3.** The fractal SG_3 is obtained from an IFS consisting of 6 contraction mappings, each with scaling ratio $\frac{1}{3}$, as indicated in Figure 3. The details of

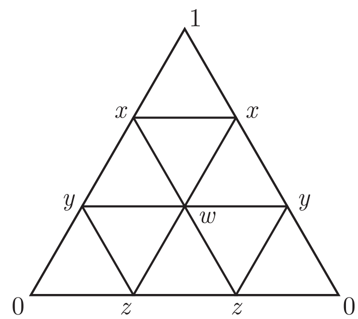

FIGURE 4. The eigenfunction extension on one $(m-1)$-cell of SG_3 to m-cells. The values on $(m-1)$-cell are $1, 0, 0$.

the spectral decimation method for SG_3 have been worked out independently in [BCD+08, DS07, Zho09a, Zho09b]. Note that p_0 is contained in three 1-cells of SG_3, in contrast to each of the other points p_i of $V_1 \setminus V_0$, which are contained in two. For this reason, we define the graph Laplacian on SG_3 as

$$\Delta_m u(x) = \frac{1}{deg(x)} \sum_{y \sim_m x} (u(y) - u(x)), \tag{6.1}$$

where $deg(x)$ is the number of m-cells containing x. From [Str06, §4.4], we have

$$\Delta_\mu u(x) = \lim_{m\to\infty} r^{-m} \left(\int_K h_x^{(m)}\, d\mu \right)^{-1} deg(x) \Delta_m u(x). \tag{6.2}$$

The renormalization constant r will be computed in §6.3.

Let p_x denote a vertex where u takes the value x as depicted in Figure 4, then by (6.1), the symmetric eigenvalue equations on V_m are

$$\begin{aligned}
\Delta_m u(p_x) &= \tfrac{1}{4}\left[(1-x)+(x-x)+(w-x)+(y-x)\right] = -\lambda'_m x \\
\Delta_m u(p_y) &= \tfrac{1}{4}\left[(x-y)+(w-y)+(z-y)+(0-y)\right] = -\lambda'_m y \\
\Delta_m u(p_z) &= \tfrac{1}{4}\left[(y-z)+(w-z)+(z-z)+(0-z)\right] = -\lambda'_m z \\
\Delta_m u(p_w) &= \tfrac{1}{6}\left[2(x-w)+2(y-w)+2(z-w)\right] = -\lambda'_m w,
\end{aligned}$$

which can be rewritten, using $\lambda_m = 4\lambda'_m$, as

$$\begin{aligned}
(4-\lambda_m)x &= 1 + x + y + w & \qquad (4-\lambda_m)y &= x + z + w \\
(4-\lambda_m)z &= y + z + w & \qquad (4-\lambda_m)w &= \tfrac{4}{3}(x+y+z).
\end{aligned}$$

For now, we suppress the dependence on m for convenience and denote $\lambda = \lambda_m$. Solving for λ, we obtain

$$x = \alpha(\lambda) := (96 - 109\lambda + 33\lambda^2 - 3\lambda^3)/\varphi(\lambda), \tag{6.3a}$$

$$y = \beta(\lambda) := (16 - 3\lambda)(3-\lambda)/\varphi(\lambda), \tag{6.3b}$$

$$z = \gamma(\lambda) := (36 - 7\lambda)/\varphi(\lambda), \tag{6.3c}$$

$$w = \rho(\lambda) := 4(5-\lambda)(3-\lambda)/\varphi(\lambda), \tag{6.3d}$$

$$\text{where} \quad \varphi(\lambda) := 3(5-\lambda)(3-\lambda)(4 - 6\lambda + \lambda^2), \tag{6.3e}$$

and we see that the forbidden eigenvalues are $3, 5, 3 \pm \sqrt{5}$.

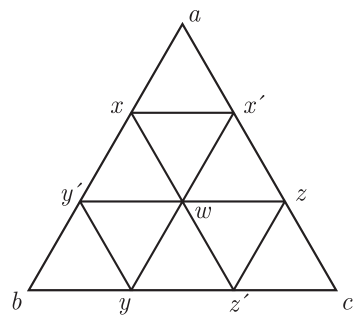

Figure 5. The labeling for a general eigenfunction extension, from one $(m-1)$-cell to six m-cells. The values on the boundary of the $(m-1)$-cell are a, b, c.

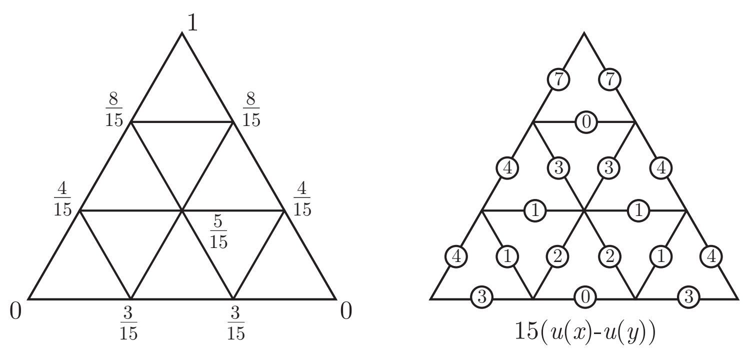

Figure 6. The harmonic extension of u on SG_3, where $u|_{V_0} = [1, 0, 0]$.

For a general function on SG_3, we extend the eigenfunction using the labeling indicated in Figure 5, as follows:

$$\begin{aligned} x &= a\alpha(\lambda) + b\beta(\lambda) + c\gamma(\lambda) & x' &= a\alpha(\lambda) + c\beta(\lambda) + b\gamma(\lambda) \\ y &= b\alpha(\lambda) + c\beta(\lambda) + a\gamma(\lambda) & y' &= b\alpha(\lambda) + a\beta(\lambda) + c\gamma(\lambda) \\ z &= c\alpha(\lambda) + a\beta(\lambda) + b\gamma(\lambda) & z' &= c\alpha(\lambda) + b\beta(\lambda) + a\gamma(\lambda) \\ w &= (a + b + c)\rho(\lambda). \end{aligned} \tag{6.4}$$

The eigenfunction extension matrix for SG_3 corresponding to F_0 is

$$A_0(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ \alpha(\lambda) & \beta(\lambda) & \gamma(\lambda) \\ \alpha(\lambda) & \gamma(\lambda) & \beta(\lambda) \end{bmatrix},$$

where we have $\alpha(\lambda), \beta(\lambda), \gamma(\lambda), \varphi(\lambda)$ as before, that is $A_0(\lambda)u|_{V_0} = u|_{F_0 V_0}$.

6.2. **Eigenfunctions of the Laplacian on** SG_3**.** Let u be a function taking values $1, 0, 0$ on V_0. The harmonic extension $\tilde{u}$ on V_1 corresponds to taking $\lambda = 0$ in the system (6.3) above, so that

$$x = \frac{8}{15}, \quad y = \frac{4}{15}, \quad z = \frac{3}{15}, \quad w = \frac{5}{15},$$

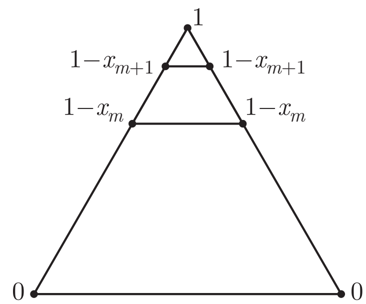

FIGURE 7. The values of the eigenfunction u on SG_3, where $u|_{V_0} = [1,0,0]^T$. This figure shows a closeup of u near the point where it takes the value 1. By symmetry, we define $x_m := 1 - u(F_0^m q_1) = 1 - u(F_0^m q_2)$.

and we have Figure 6. Following [Str06, §1.3], the energy renormalization constant computed by

$$E_1(\tilde{u}) = \left(\frac{1}{15}\right)^2 \left(4\cdot 1^2 + 2\cdot 2^2 + 4\cdot 3^2 + 4\cdot 4^2 + 2\cdot 7^2\right) = \frac{14}{15}.$$

Since $E_0(u) = 1 + 1 = 2$,

$$2 = E_0(u) = r^{-1}E_1(\tilde{u}) = r^{-1}\frac{14}{15} \qquad \Longrightarrow \qquad r = \frac{7}{15}. \tag{6.5}$$

Thus, the normal derivatives on SG_3 are computed by

$$\partial_n u(p) = \lim_{m\to\infty}\left(\frac{15}{7}\right)^m \sum_{p\sim_m y}(u(p) - u(y)). \tag{6.6}$$

Theorem 6.1. *The pointwise formulation of the Laplacian on SG_3 is*

$$\Delta_\mu u(x) = 6\lim_{m\to\infty}\left(\frac{90}{7}\right)^m \Delta_m u(x_m), \tag{6.7}$$

where $\{x_m\}$ is any sequence with $\lim x_m = x$ and $x_m \in V_m$.

Proof. Following [Str06, §2.2], it is easy to compute

$$\int h_{x_m}^{(m)}\,d\mu = \begin{cases} \frac{2}{3\cdot 6^m} & if\ \ deg(x_m) = 4, \\ \frac{1}{6^m} & if\ \ deg(x_m) = 6 \end{cases}$$

since μ is the standard (self-similar) measure on SG_3. Thus, by (6.2),

$$\Delta_\mu u(x) = \lim_{m\to\infty}\left(\frac{15}{7}\right)^m \cdot 6^{m+1}\Delta_m u(x) = 6\lim_{m\to\infty}\left(\frac{90}{7}\right)^m \Delta_m u(x_m), \tag{6.8}$$

□

Throughout, whenever there is discussion of an eigenvalue λ, we assume that we have been given the sequence $\{\lambda_m\}_{m=0}^\infty$ which defines λ via the decimation formula. Thus by Theorem 6.1,

$$\lambda = 6\lim_{m\to\infty}\left(\frac{90}{7}\right)^m \lambda'_m = \frac{3}{2}\lim_{m\to\infty}\left(\frac{90}{7}\right)^m \lambda_m. \tag{6.9}$$

6.3. **Computation of the normal derivatives.**

Theorem 6.2. *Let $-\Delta u = \lambda u$ on SG_3, where u is defined on V_0 by $u(q_0) = 1$, $u(q_1) = 0$, and $u(q_2) = 0$. Define*

$$\tau(\lambda) := \frac{2\lambda}{3\lambda_0} \prod_{j=1}^{\infty} \frac{(1 - \frac{\lambda_j}{4})(1 - \frac{\lambda_j}{6})}{1 - \frac{3}{2}\lambda_j + \frac{\lambda_j^2}{4}}. \tag{6.10}$$

Then the normal derivatives of u are

$$\partial_n u(q_0) = \frac{4 - \lambda_0}{2}\tau(\lambda), \quad \textit{and} \tag{6.11a}$$

$$\partial_n u(q_1) = \partial_n u(q_2) = -\tau(\lambda). \tag{6.11b}$$

Proof. To obtain (6.11a) we need the values $u(F_0^m q_1) = u(F_0^m q_2) = 1 - x_m$ as depicted in Figure 7.

Claim: $u(F_0^m q_1) = u(F_0^m q_2) = 1 - x_m$, where $x_0 = 1$ and

$$x_{m+1} - \frac{\lambda_{m+1}}{4} = \frac{(4 - \lambda_{m+1})(6 - \lambda_{m+1})\lambda_{m+1}}{(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)\lambda_m}\left(x_m - \frac{\lambda_m}{4}\right). \tag{6.12}$$

Proof of claim. By (6.4), if $b = c$, then $x = x'$. Then from $u(q_1) = u(q_2) = 0$, we will have $u(F_0^m q_1) = u(F_0^m q_2)$ for all m, by induction. Define $x_m := 1 - u(F_0^m q_1)$, $m \geq 0$. From $u(q_1) = 0$, we have $x_0 = 1$. Now we show (6.12) holds.

Denote $\delta(\lambda) := \beta(\lambda) + \gamma(\lambda)$, so that $\delta(\lambda) = (14 - 3\lambda)(6 - \lambda)/\varphi(\lambda)$, where $\varphi(\lambda)$ is as in (6.3). Using (6.4), we have the matrix equation

$$A_0(\lambda)\begin{bmatrix} 1 \\ 1 - x_m \\ 1 - x_m \end{bmatrix} = \begin{bmatrix} 1 \\ 1 - x_{m+1} \\ 1 - x_{m+1} \end{bmatrix}$$

gives

$$\begin{aligned} x_{m+1} &= 1 - \alpha(\lambda_{m+1}) - \delta(\lambda_{m+1}) + \delta(\lambda_{m+1})x_m \\ &= -\frac{\lambda_{m+1}(5 - \lambda_{m+1})}{4 - 6\lambda_{m+1} + \lambda_{m+1}^2} + \frac{(14 - 3\lambda_{m+1})(6 - \lambda_{m+1})}{\varphi(\lambda_{m+1})}x_m. \end{aligned}$$

From the decimation relation [DS07, (2.12)], we have the identity

$$\frac{3(5 - \lambda_{m+1})(4 - \lambda_{m+1})(3 - \lambda_{m+1})\lambda_{m+1}}{(14 - 3\lambda_{m+1})\lambda_m} = 1,$$

so that

$$\begin{aligned} \delta(\lambda_{m+1}) &= \delta(\lambda_{m+1})\frac{3(5 - \lambda_{m+1})(4 - \lambda_{m+1})(3 - \lambda_{m+1})\lambda_{m+1}}{(14 - 3\lambda_{m+1})\lambda_m} \\ &= \frac{(4 - \lambda_{m+1})(6 - \lambda_{m+1})\lambda_{m+1}}{(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)\lambda_m}. \end{aligned}$$

We would like to see $x_{m+1} - f(\lambda_{m+1}) = \delta(\lambda_{m+1})(x_m - f(\lambda_m))$ for some function f, which is equivalent to

$$(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)\frac{f(\lambda_{m+1})}{\lambda_{m+1}} = \frac{f(\lambda_m)}{\lambda_m}(4 - \lambda_{m+1})(6 - \lambda_{m+1}) - (5 - \lambda_{m+1}).$$

Let $f(x) = xg(x)$ and this can be rewritten

$$(4 - 6\lambda_{m+1} + \lambda_{m+1}^2)g(\lambda_{m+1}) = g(\lambda_m)(24 - 10\lambda_{m+1} + \lambda_{m+1}^2) - (5 - \lambda_{m+1}),$$

which is easily seen to be true for the constant function $g(x) = \frac{1}{4}$. Hence we may define $f(x) = \frac{x}{4}$, to obtain

$$x_{m+1} - \frac{\lambda_{m+1}}{4} = \delta(\lambda_{m+1})\left(x_m - \frac{\lambda_m}{4}\right). \qquad \square$$

Now we compute $\partial_{\mathrm{n}} u(q_0)$ using (6.12) to obtain

$$x_m - \frac{\lambda_m}{4} = \left(1 - \frac{\lambda_0}{4}\right)\frac{\lambda_m}{\lambda_0}\prod_{j=1}^{m}\frac{(4-\lambda_j)(6-\lambda_j)}{4-6\lambda_j+\lambda_j^2}$$

$$x_m = \frac{4-\lambda_0}{4\lambda_0}\left(\frac{\lambda_0}{4-\lambda_0} + \prod_{j=1}^{m}\frac{(4-\lambda_j)(6-\lambda_j)}{4-6\lambda_j+\lambda_j^2}\right)\lambda_m.$$

Since $u(q_0) = 1$, we apply (6.6) to compute

$$\begin{aligned}
\partial_{\mathrm{n}} u(q_0) &= \lim_{m\to\infty}\left(\frac{15}{7}\right)^m\left(\frac{6}{6}\right)^m(2u(q_0) - 2(1-x_m)) \\
&= \frac{4-\lambda_0}{2\lambda_0}\lim_{m\to\infty}\left(\frac{90}{7}\right)^m\lambda_m\left(\frac{\lambda_0}{6^m(4-\lambda_0)} + \prod_{j=1}^{m}\frac{(4-\lambda_j)(6-\lambda_j)}{6(4-6\lambda_j+\lambda_j^2)}\right) \\
&= \frac{4-\lambda_0}{2\lambda_0}\left(\frac{2}{3}\lambda\right)\left(0 + \prod_{j=1}^{\infty}\frac{(4-\lambda_j)(6-\lambda_j)}{6(4-6\lambda_j+\lambda_j^2)}\right),
\end{aligned}$$

which is equivalent to the result.

Now we compute the normal derivatives (6.11a). To obtain $\partial_{\mathrm{n}} u(q_1) = \partial_{\mathrm{n}} u(q_2)$, we don't actually need the values $u(F_1^m q_0)$ and $u(F_1^m q_2)$ as depicted in Figure 8. Instead, it suffices to only compute their sum, since by (6.6), one has

$$\partial_{\mathrm{n}} u(q_1) = \partial_{\mathrm{n}} u(q_2) = -\lim_{m\to\infty}\left(\frac{15}{7}\right)^m\left(u(F_1^m q_0) + u(F_1^m q_2)\right). \tag{6.13}$$

To exploit this symmetry accordingly, define

$$y_m := u(F_0^m q_1), \quad z_m := u(F_0^m q_2), \quad \text{and} \quad s_m := y_m + z_m.$$

Claim: the sequence $\{s_m\}_{m=0}^{\infty}$ is given recurrently by $s_0 = 1$ and

$$s_{m+1} = \frac{(14-3\lambda_{m+1})(6-\lambda_{m+1})}{\varphi(\lambda_{m+1})}s_m. \tag{6.14}$$

Proof of claim. As indicated in Figure 8, dihedral symmetry allows us to continue using the same matrix $A_0(\lambda)$ for computations, as long as we use $[0, 1, 0]^T$ for the new boundary data.

It is clear that $s_0 = 1 + 0$ from the values on V_0. Then using the notation $\delta(\lambda) = \alpha(\lambda) + \beta(\lambda)$ as above, the matrix equation

$$A_0(\lambda)\begin{bmatrix} 0 \\ y_m \\ z_m \end{bmatrix} = \begin{bmatrix} 0 \\ \beta(\lambda_{m+1})y_m + \gamma(\lambda_{m+1})z_m \\ \gamma(\lambda_{m+1})y_m + \beta(\lambda_{m+1})z_m \end{bmatrix} = \begin{bmatrix} 0 \\ y_{m+1} \\ z_{m+1} \end{bmatrix}$$

gives $s_{m+1} = y_{m+1} + z_{m+1} = \delta(\lambda_{m+1})s_m$ immediately. $\square$

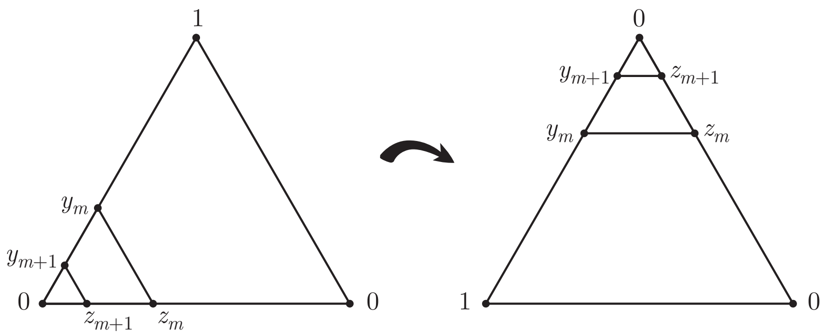

FIGURE 8. The values of the eigenfunction u on SG_3, where $u|_{V_0} = [0,1,0]^T$. This figure shows a closeup of u near a point where it takes the value 0. See (6.13) and the ensuing discussion.

Since (6.14) gives

$$s_m = \prod_{j=1}^{m} \delta(\lambda_j) s_0 = \frac{\lambda_m}{\lambda_0} \prod_{j=1}^{m} \frac{(4-\lambda_j)(6-\lambda_j)}{4-6\lambda_j+\lambda_j^2},$$

and $u(q_1) = 0$, the normal derivative is

$$\begin{aligned} \partial_{\mathrm{n}} u(q_1) &= \lim_{m\to\infty} \left(\frac{15}{7}\right)^m \left(\frac{6}{6}\right)^m (2u(q_1) - s_m) \\ &= -\frac{1}{\lambda_0} \lim_{m\to\infty} \left(\frac{90}{7}\right)^m \lambda_m \prod_{j=1}^{m} \frac{(4-\lambda_j)(6-\lambda_j)}{6(4-6\lambda_j+\lambda_j^2)} \\ &= -\frac{1}{\lambda_0} \left(\frac{2}{3}\lambda\right) \prod_{j=1}^{\infty} \frac{(4-\lambda_j)(6-\lambda_j)}{6(4-6\lambda_j+\lambda_j^2)}. \end{aligned}$$

□

6.4. **The resolvent prekernel.** As in (1.16), let $B_{pq}^{(\lambda)} := \sum_{K_j \ni q} \partial_{\mathrm{n}}^{K_j} \psi_\lambda^{(p)}(q)$ for $p \in V_1 \setminus V_0$.

Corollary 6.3. *With $\tau(\lambda)$ as in Thm. 6.2 and $r = \frac{7}{15}$,*

$$B_{pq}^{(\lambda)} = -r^{-1}\tau(\lambda), \text{ for } p \sim_1 q \quad \text{and} \quad B_{pp}^{(\lambda)} = \begin{cases} \frac{3}{2} r^{-1}(4-\lambda_0)\tau(\lambda), & p = p_0 \\ r^{-1}(4-\lambda_0)\tau(\lambda), & p \neq p_0. \end{cases}$$

Proof. We are now working on V_1, so each term has a leading factor of r^{-1}. Whenever $p \sim_1 q$, there is just one term $\partial_{\mathrm{n}} u(q) = -\tau(\lambda)$ in the sum; the other corner of the triangle is ignored and everything outside this 1-cell is 0. When $p = q$, then there is a sum of terms $\partial_{\mathrm{n}} u(p) = \frac{4-\lambda_0}{2}\tau(\lambda)$. At the center point p_0, there are three such terms; at every other point there are only two. □

The matrix $B^{(\lambda)}_{pq}$ is

$$\frac{15}{7}\tau(\lambda)\begin{bmatrix} \frac{3}{2}(4-\lambda_0) & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & (4-\lambda_0) & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & (4-\lambda_0) & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & (4-\lambda_0) & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & (4-\lambda_0) & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & (4-\lambda_0) & -1 \\ -1 & -1 & 0 & 0 & 0 & -1 & (4-\lambda_0) \end{bmatrix} \tag{6.15}$$

Definition 6.4. Define the *resolvent prekernel* by $G^{(\lambda)} := (B^{(\lambda)})^{-1}$.

Our final result may be obtained by brutal and direct computation.

Theorem 6.5. *The resolvent prekernel $G^{(\lambda)}$ is given by*

$$\frac{14}{15(6-\lambda)\tau(\lambda)\varphi(\lambda)}\begin{bmatrix} (2-\lambda)\kappa_1 & \kappa_1 & \kappa_1 & \kappa_1 & \kappa_1 & \kappa_1 & \kappa_1 \\ \kappa_1 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_4 & \kappa_3 \\ \kappa_1 & \kappa_3 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_4 \\ \kappa_1 & \kappa_4 & \kappa_3 & \kappa_2 & \kappa_3 & \kappa_4 & \kappa_5 \\ \kappa_1 & \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 & \kappa_3 & \kappa_4 \\ \kappa_1 & \kappa_4 & \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 & \kappa_3 \\ \kappa_1 & \kappa_3 & \kappa_4 & \kappa_5 & \kappa_4 & \kappa_3 & \kappa_2 \end{bmatrix}, \tag{6.16}$$

where

$$\begin{aligned} \kappa_1 &= (3-\lambda)(5-\lambda)(6-\lambda), \\ \kappa_2 &= 201 - 300\lambda + \tfrac{269}{2}\lambda^2 - 24\lambda^3 + \tfrac{3}{2}\lambda^4, \\ \kappa_3 &= 87 - 75\lambda + 19\lambda^2 - \tfrac{3}{2}\lambda^3, \\ \kappa_4 &= 57 - 24\lambda + \tfrac{5}{2}\lambda^2, \text{ and} \\ \kappa_5 &= 51 - 15\lambda - \lambda^2. \end{aligned}$$

In particular, $G^{(\lambda)}$ is symmetric and invertible with determinant

$$\det G^{(\lambda)} = \left(\frac{7}{15}\right)^7 \frac{6(4-6\lambda+\lambda^2)}{(6-\lambda)\varphi(\lambda)^2\tau(\lambda)}. \tag{6.17}$$

References

[BCD+08] Neil Bajorin, Tao Chen, Alon Dagan, Catherine Emmons, Mona Hussein, Michael Khalil, Poorak Mody, Benjamin Steinhurst, and Alexander Teplyaev. Vibration modes of 3*n*-gaskets and other fractals. *J. Phys. A: Math. Theor.*, 41:015101 (21pp), 2008.

[DRS09] Jessica L. DeGrado, Luke G. Rogers, and Robert S. Strichartz. Gradients of Laplacian eigenfunctions on the Sierpinski gasket. *Proc. Amer. Math. Soc.*, (137):531–540, 2009.

[DS07] Sean Drenning and Robert S. Strichartz. Spectral decimation on Hambly's homogeneous hierarchical gaskets. pages 1–23, 2007.

[FHK94] Pat J. Fitzsimmons, Ben M. Hambly, and Takashi Kumagai. Transition density estimates for Brownian motion on affine nested fractals. *Comm. Math. Phys.*, 165(3):595–620, 1994.

[HK99] Ben M. Hambly and Takashi Kumagai. Transition density estimates for diffusion processes on post critically finite self-similar fractals. *Proc. London Math. Soc. (3)*, 78(2):431–458, 1999.

[Hut81] John E. Hutchinson. Fractals and self-similarity. *Indiana Univ. Math. J.*, 30(5):713–747, 1981.

[Kig01] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[Kig03] Jun Kigami. Harmonic analysis for resistance forms. *J. Funct. Anal.*, 204(2):399–444, 2003.

[Rog08] Luke Rogers. The resolvent kernel for pcf self-similar fractals, part ii. *in preparation*, 2008.

[Sab97] Christophe Sabot. Existence and uniqueness of diffusions on finitely ramified self-similar fractals. *Ann. Sci. École Norm. Sup. (4)*, 30(5):605–673, 1997.

[See67] Robert T. Seeley. Complex powers of an elliptic operator. In *Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966)*, pages 288–307. Amer. Math. Soc., Providence, R.I., 1967.

[See69] Robert T. Seeley. The resolvent of an elliptic boundary problem. *Amer. J. Math.*, 91:889–920, 1969.

[Str06] Robert S. Strichartz. *Differential equations on fractals*. Princeton University Press, Princeton, NJ, 2006. A tutorial.

[Tep98] Alexander Teplyaev. Spectral analysis on infinite Sierpiński gaskets. *J. Funct. Anal.*, 159(2):537–567, 1998.

[Zho09a] Denglin Zhou. Criteria for spectral gaps of laplacians on fractals. *J. Fourier. Anal. Appl.*, (to appear), 2009.

[Zho09b] Denglin Zhou. Spectral analysis of Laplacians on the Vicsek set. *Pac. J. Math.*, (to appear), 2009.

Cornell University, Ithaca, NY 14850-4201 USA
E-mail address: `mionescu@math.cornell.edu`

University of Iowa, Iowa City, IA 52246-1419 USA
E-mail address: `erin-pearse@uiowa.edu`

University of Connecticut, Storrs, CT 06269-3009 USA
E-mail address: `rogers@math.uconn.edu`

Zhejiang University, Hangzhou, 310027, China, and Cornell University, Ithaca, NY 14850-4201 USA
E-mail address: `ruanhj@zju.edu.cn`

Cornell University, Ithaca, NY 14850-4201 USA
E-mail address: `str@math.cornell.edu`